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Chapter 1

Collected Articles on Inequalities

After reading the previous chapters, you should have gained a lot of insight into inequalities. The world of inequalities is really uniquely wonderful and interesting to explore. In this chapter, we will examine inequalities in a more general and larger context with the help of the mathematical techniques and methods developed in the previous chapter.

This chapter contains 19 sections, organized into 8 articles. Many interesting matters will be discussed here, such as some generalizations of **Schur** inequality, some estimations of familiar expressions, some strange kinds of inequalities, some improvements of the classical mixing variable method and some applications of **Karamata** inequality. We wish to receive more comments and contributions from you, the readers.

Article 1

Generalization of Schur Inequality

1.1 Generalized Schur Inequality for Three Numbers

We will be talking about **Schur** inequality in these pages. Just **Schur** inequality? And is it really necessary to review it now? Yes, certainly! But instead of using **Schur** inequality in "brute force" solutions (eg. solutions that use long, complicated expanding), we will discover a very simple generalization of **Schur** inequality. An eight-grade student can easily understand this matter; however, its wide and effective influence may leave you surprised.

Theorem 1 (Generalized Schur Inequality)**.** *Let a, b, c, x, y, z be six non-negative real numbers such that the sequences (a, b, c) and (x, y, z) are monotone, then*

$$x(a-b)(a-c)+y(b-a)(b-c)+z(c-a)(c-b) \geq 0.$$

PROOF. WLOG, assume that $a \geq b \geq c$. Consider the following cases

(i). $x \geq y \geq z$. Then, we have $(c-a)(c-b) \geq 0$, so $z(c-a)(c-b) \geq 0$. Moreover,

$$x(a-c)-y(b-c) \geq x(b-c)-y(b-c)=(x-y)(b-c) \geq 0$$

$$\Rightarrow x(a-b)(a-c)+y(b-a)(b-c) \geq 0.$$

Summing up these relations, we have the desired result.

(i). $x \leq y \leq z$. We have $(a-b)(a-c) \geq 0$, so $x(a-b)(a-c) \geq 0$. Moreover,

$$z(a-c)-y(a-b) \geq z(a-b)-y(a-b)=(z-y)(a-b) \geq 0$$

$$\Rightarrow z(c-a)(c-b)+y(b-a)(b-c) \geq 0.$$

Summing up the inequalities above, we have the desired result.

Comment. Denote $S=\sum_{cyc} x(a-b)(a-c)$. By the same reasoning as above, we can prove that $S \geq 0$ if at least one of the following stronger conditions is fulfilled

1. If $a \geq b \geq c \geq 0, x \geq y \geq 0$ and $z \geq 0$.

2. If $a \geq b \geq c \geq 0, y \geq z \geq 0$ and $x \geq 0$.

3. If $a \geq b \geq c \geq 0$ and $ax \geq by \geq 0$ or $by \geq cz \geq 0$.

(1) and (2) are quite obvious. To prove (3), just notice that if $a, b, c > 0$ then

$$\frac{1}{abc}\left(x(a-b)(a-c)+y(b-a)(b-c)+z(c-a)(c-b)\right)$$
$$=ax\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)+by\left(\frac{1}{b}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{c}\right)+cz\left(\frac{1}{c}-\frac{1}{a}\right)\left(\frac{1}{c}-\frac{1}{b}\right),$$

and the problem turns to a normal form of the generalized **Schur** inequality shown above.

∇

This was such an easy proof! But you need to know that this *simple* theorem always provides unexpectedly *simple* solutions to a lot of difficult problems. That makes the difference, not its simple solution. Let's see some examples and you will understand why many inequality solvers like to use the generalized **Schur** inequality in their proofs.

Example 1.1.1. *Let a, b, c be three positive real numbers. Prove that*

$$a+b+c \leq \frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}.$$

(Ho Joo Lee)

SOLUTION. According to the identity

$$\frac{a^2+bc}{b+c}-a=\frac{(a-b)(a-c)}{b+c},$$

we can change our inequality into the form

$$x(a-b)(a-c)+y(b-a)(b-c)+z(c-a)(c-b) \geq 0,$$

in which

$$x=\frac{1}{b+c}\ ;\ y=\frac{1}{c+a}\ ;\ z=\frac{1}{a+b}\ .$$

WLOG, assume that $a \geq b \geq c$, then clearly $x \leq y \leq z$. The conclusion follows from the generalized **Schur** inequality instantly.

∇

Example 1.1.2. *Let a, b, c be positive real numbers with sum 3. Prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c} \geq \frac{3}{2a^2+bc}+\frac{3}{2b^2+ac}+\frac{3}{2c^2+ab}.$$

(Pham Kim Hung)

SOLUTION. Rewrite the inequality into the following from

$$\sum_{cyc}\left(\frac{1}{a}-\sum_{cyc}\frac{a+b+c}{2a^2+bc}\right)\geq 0 \;\Leftrightarrow\; \sum_{cyc}\frac{(a-b)(a-c)}{2a^3+abc}\geq 0.$$

Notice that if $a\geq b\geq c$ then

$$\frac{1}{2a^3+abc}\leq\frac{1}{2b^3+abc}\leq\frac{1}{2c^3+abc}.$$

The conclusion follows from the generalized **Schur** inequality.

∇

Example 1.1.3. *Let a,b,c be the side lengths of a triangle. Prove that*

$$\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}+\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}}+\frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}}\leq 3.$$

(Italian Winter Camp 2007)

SOLUTION. By a simple observation, the inequality is equivalent to

$$\sum_{cyc}\left(1-\frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\right)\geq 0 \;\Leftrightarrow\; \sum_{cyc}\frac{\sqrt{a}+\sqrt{b}-\sqrt{c}-\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}}\geq 0$$

$$\Leftrightarrow \sum_{cyc}\frac{\sqrt{ab}-\sqrt{c(a+b-c)}}{\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{a+b-c}\right)}\geq 0$$

$$\Leftrightarrow \sum_{cyc}\frac{(c-a)(c-b)}{S_c}\geq 0,$$

where

$$S_c=\left(\sqrt{a}+\sqrt{b}-\sqrt{c}\right)\left(\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{a+b-c}\right)\left(\sqrt{ab}+\sqrt{c(a+b-c)}\right),$$

and S_a, S_b are determined similarly. It's easy to check that if $b\geq c$ then

$$\sqrt{a}+\sqrt{b}-\sqrt{c}\geq\sqrt{c}+\sqrt{a}-\sqrt{b}\,;$$

$$\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{a+b-c}\geq\sqrt{a}+\sqrt{b}+\sqrt{c}+\sqrt{c+a-b}\,;$$

$$\sqrt{ab}+\sqrt{c(a+b-c)}\geq\sqrt{ca}+\sqrt{b(c+a-b)}\,;$$

Therefore $S_c\geq S_b$. The proof is finished by the generalized **Schur** inequality.

∇

Example 1.1.4. *Let x, y, z be positive real numbers such that $\sqrt{x}+\sqrt{y}+\sqrt{z}=1$. Prove that*

$$\frac{x^2+yz}{x\sqrt{2(y+z)}}+\frac{y^2+zx}{y\sqrt{2(z+x)}}+\frac{z^2+xy}{z\sqrt{2(x+y)}}\geq 1.$$

(APMO 2007)

SOLUTION. We use the following simple transformation

$$\sum_{cyc}\frac{x^2+yz}{x\sqrt{2(y+z)}}=\sum_{cyc}\frac{(x-y)(x-z)+x(y+z)}{x\sqrt{2(y+z)}}$$
$$=\sum_{cyc}\frac{(x-y)(x-z)}{x\sqrt{2(y+z)}}+\sum_{cyc}\sqrt{\frac{y+z}{2}}.$$

By the generalized **Schur** inequality, we get that

$$\sum_{cyc}\frac{(x-y)(x-z)}{x\sqrt{2(y+z)}}\geq 0.$$

By **AM-GM** inequality, the remaining work is obvious

$$\sum_{cyc}\sqrt{\frac{y+z}{2}}\geq\sum_{cyc}\frac{1}{2}\left(\sqrt{y}+\sqrt{z}\right)=\sum_{cyc}\sqrt{x}=1.$$

This ends the proof. Equality holds for $x=y=z=\frac{1}{9}$.

∇

Example 1.1.5. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2+2bc}{(b+c)^2}+\frac{b^2+2ac}{(a+c)^2}+\frac{c^2+2ab}{(a+b)^2}\geq\frac{9}{4}.$$

SOLUTION. The inequality can be rewritten as

$$\sum_{cyc}\frac{(a-b)(a-c)+(ab+bc+ca)}{(b+c)^2}\geq\frac{9}{4},$$

or equivalently $A+B\geq\frac{9}{4}$, where

$$A=\frac{(a-b)(a-c)}{(b+c)^2}\ ;\ B=\sum\frac{ab+bc+ca}{(b+c)^2}\ ;$$

By the generalized **Schur** inequality, we deduce that $A\geq 0$. Moreover, $B\geq\frac{9}{4}$ by **Iran 96** inequality. Therefore, we are done and the equality holds for $a=b=c$.

∇

Example 1.1.6. *Let a, b, c be positive real numbers. Prove that*

$$\sqrt{\frac{a^3+abc}{(b+c)^3}}+\sqrt{\frac{b^3+abc}{(c+a)^3}}+\sqrt{\frac{c^3+abc}{(a+b)^3}} \geq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}.$$

(Nguyen Van Thach)

SOLUTION. Notice that

$$\sqrt{\frac{a^3+abc}{(b+c)^3}}-\frac{a}{b+c}=\frac{\sqrt{a}}{b+c}\cdot\left(\sqrt{\frac{a^2+bc}{b+c}}-\sqrt{a}\right)$$

$$=\frac{\sqrt{a}(a-b)(a-c)}{(b+c)\sqrt{b+c}\left(\sqrt{a^2+bc}+\sqrt{a(b+c)}\right)}.$$

The inequality can be rewritten as $\sum_{cyc} S_a(a-b)(a-c) \geq 0$ with

$$S_a=\frac{\sqrt{a}}{(b+c)\sqrt{b+c}\left(\sqrt{a^2+bc}+\sqrt{a(b+c)}\right)} \;;$$

$$S_b=\frac{\sqrt{b}}{(c+a)\sqrt{c+a}\left(\sqrt{c^2+ab}+\sqrt{c(a+b)}\right)} \;;$$

$$S_c=\frac{\sqrt{c}}{(a+b)\sqrt{a+b}\left(\sqrt{c^2+ab}+\sqrt{c(a+b)}\right)} \;.$$

Now suppose that $a \geq b \geq c$, then it's easy to get that

$$(b+c)\sqrt{a^2+bc} \leq (a+c)\sqrt{b^2+ac} \;;$$

$$(b+c)\sqrt{a(b+c)} \leq (a+c)\sqrt{b(a+c)} \;.$$

Thus $(b+c)\left(\sqrt{a^2+bc}+\sqrt{a(b+c)}\right) \geq (c+a)\left(\sqrt{c^2+ab}+\sqrt{c(a+b)}\right)$ and therefore we have $S_a \geq S_b$. We can conclude that

$$\sum_{cyc} S_a(a-b)(a-c) \geq S_a(a-b)(a-c)+S_b(b-a)(b-c)$$

$$\geq (S_a-S_b)(a-b)(b-c) \geq 0.$$

This is the end of the proof. The equality holds for $a=b=c$.

∇

Example 1.1.7. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a^2}{(2a+b)(2a+c)}+\frac{b^2}{(2b+c)(2b+a)}+\frac{c^2}{(2c+a)(2c+b)} \leq \frac{1}{3}.$$

SOLUTION. If $c = 0$, the problem is obvious. Suppose that $a, b, c > 0$, then we have

$$1 - 3\sum_{cyc}\frac{a^2}{(2a+b)(2a+c)} = \sum_{cyc}\left(\frac{a}{a+b+c} - \frac{a^2}{(2a+b)(2a+c)}\right)$$
$$= \sum_{cyc}\frac{a(a-b)(a-c)}{(a+b+c)(2a+b)(2a+c)}$$
$$= \sum_{cyc}\frac{a^2\left(\frac{1}{a}-\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{c}\right)}{(a+b+c)(2a+b)(2a+c)}.$$

By the generalized **Schur** inequality, it suffices to prove that if $a \geq b$ then

$$\frac{a^2}{(2a+b)(2a+c)} \geq \frac{b^2}{(2b+a)(2b+c)}.$$

But the previous inequality is equivalent to

$$(a-b)\left(2ab(a+b+c) + c(a^2+ab+b^2)\right) \geq 0,$$

which is obvious. The equality holds for $a = b = c$ and $a = b, c = 0$ up to permutation.

∇

Example 1.1.8. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{a^2+2bc} + \frac{1}{b^2+2ca} + \frac{1}{c^2+2ab} \leq \left(\frac{a+b+c}{ab+bc+ca}\right)^2.$$

(Pham Huu Duc)

SOLUTION. We have

$$\sum_{cyc}\frac{ab+bc+ca}{a^2+2bc} - \frac{(a+b+c)^2}{ab+bc+ca} = \sum_{cyc}\left(\frac{ab+bc+ca}{a^2+2bc} - 1\right) + \sum_{cyc}\frac{(c-a)(c-b)}{ab+bc+ca}$$
$$= \sum_{cyc}(a-b)(a-c)\left(\frac{1}{a^2+2bc} + \frac{1}{ab+bc+ca}\right).$$

WLOG, assume that $a \geq b \geq c$. By the generalized **Schur** inequality, it suffices to prove that

$$a\left(\frac{1}{a^2+2bc} + \frac{1}{ab+bc+ca}\right) \geq b\left(\frac{1}{b^2+2ca} + \frac{1}{ab+bc+ca}\right).$$

Indeed, the difference between the left-hand side and the right-hand side is

$$\frac{a-b}{ab+bc+ca} - \frac{(a-b)(2ca+2cb-ab)}{(a^2+2bc)(b^2+2ca)}$$

$$= \frac{(a-b)\left(2a^2b^2 + c(a^3+b^3) - 2a^2c^2 - 2b^2c^2 + c(a-b)^2(a+b)\right)}{(ab+bc+ca)(a^2+2bc)(b^2+2ca)} \geq 0.$$

We just got the desired result. The equality holds for $a = b = c$.

∇

Example 1.1.9. *Let a, b, c be non-negative real numbers such that $ab + bc + ca = 1$. Prove that*

$$\frac{1+b^2c^2}{(b+c)^2} + \frac{1+c^2a^2}{(c+a)^2} + \frac{1+a^2b^2}{(a+b)^2} \geq \frac{5}{2}.$$

(Mathlinks Contest)

SOLUTION. First we have that

$$\sum_{cyc} \frac{1+b^2c^2}{(b+c)^2} = \sum_{cyc} \frac{(ab+bc+ca)^2 + b^2c^2}{(b+c)^2} = \sum_{cyc} a^2 + 2\sum_{cyc} \frac{abc}{b+c} + \frac{2b^2c^2}{(b+c)^2}.$$

Therefore, our inequality can be rewritten as

$$2\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) + \sum_{cyc} a\left(\frac{4bc}{b+c} - (b+c)\right) + \sum_{cyc}\left(\frac{4b^2c^2}{(b+c)^2} - bc\right) \geq 0.$$

The left hand expression of the previous inequality is

$$\sum_{cyc} \frac{(b^2+bc+c^2-ab-ac)(b-c)^2}{(b+c)^2} = \sum_{cyc} \frac{(b-c)\left(b^3-c^3-a(b^2-c^2)\right)}{(b+c)^2}$$

$$= \sum_{cyc} \frac{(b-c)\left(b^2(b-a) - c^2(c-a)\right)}{(b+c)^2} = \sum_{cyc}(b-c)(b-a)\left(\frac{b^2}{(b+c)^2} + \frac{b^2}{(b+a)^2}\right),$$

and the proof is completed by the generalized **Schur** inequality because if $b \geq c$ then

$$\frac{b^2}{(b+c)^2} + \frac{b^2}{(b+a)^2} \geq \frac{c^2}{(c+a)^2} + \frac{c^2}{c+b)^2}.$$

∇

1.2 A Generalization of Schur Inequality for n Numbers

If **Schur** inequality for three variables and its generalized form have been discussed thoroughly in the previous section, we now go ahead to the generalization of **Schur** inequality for n variables. As a matter of fact, we want an estimation of

$$F_n = a_1(a_1-a_2)...(a_1-a_n)+a_2(a_2-a_1)(a_2-a_3)...(a_2-a_n)+...+a_n(a_n-a_1)...(a_n-a_{n-1}).$$

The first question is if the inequality $F_n \geq 0$ holds. Unfortunately, it is not always true (it is only true for $n = 3$). Furthermore, the general inequality

$$a_1^k(a_1-a_2)...(a_1-a_n)+a_2^k(a_2-a_1)(a_2-a_3)...(a_2-a_n)+...+a_n^k(a_n-a_1)...(a_n-a_{n-1}) \geq 0$$

is also false for all $n \geq 4$ and $k \geq 0$. To find a counter-example, we have to check the case $n = 4$ only and notice that if $n > 4$, we can choose $a_k = 0\ \forall k \geq 4$. For $n = 4$, consider the inequality

$$a^k(a-b)(a-c)(a-d)+b^k(b-a)(b-c)(b-d)+c^k(c-a)(c-b)(c-d)+d^k(d-a)(d-b)(d-c) \geq 0.$$

Just choose $a = b = c$, the inequality becomes $d^k(d-a)^3 \geq 0$, which is clearly false (when $d \leq a$).

Our work now is to find another version of this inequality. To do so, we first have to find something new in the simple case $n = 4$. The following results are quite interesting.

Example 1.2.1. *Let a, b, c, d be non-negative real numbers such that $a + b + c + d = 1$. Prove that*

$$a(a-b)(a-c)(a-d)+b(b-a)(b-c)(b-d)+c(c-a)(c-b)(c-d)+d(d-a)(d-b)(d-c) \geq \frac{-1}{432}.$$

(Pham Kim Hung)

SOLUTION. We use the entirely mixing variable and the renewed derivative to solve this problem. Notice that our inequality is exactly

$$\frac{1}{432}(a+b+c+d)^4 + \sum_{cyc} a(a-b)(a-c)(a-d) \geq 0.$$

Notice that the inequality is clearly true if $d = 0$, so we only need to prove that (after taking the global derivative)

$$\frac{1}{27}(a+b+c+d)^3 + \sum_{cyc} (a-b)(a-c)(a-d) \geq 0.$$

Because the expression

$$\sum_{cyc}(a-b)(a-c)(a-d)$$

is unchanged if we decrease a, b, c all at once, it suffices to consider the inequality in case $\min\{a, b, c, d\} = 0$. WLOG, assume that $d = 0$, then the inequality becomes

$$\frac{1}{27}(a+b+c)^3 + a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) - abc \geq 0.$$

This inequality follows from **AM-GM** inequality and **Schur** inequality immediately. We are done. Equality cannot hold.

∇

Example 1.2.2. *Let a, b, c, d be non-negative real numbers. Prove that*

$$a(a-b)(a-c)(a-d)+b(b-a)(b-c)(b-d)+c(c-a)(c-b)(c-d)+d(d-a)(d-b)(d-c)+abcd \geq 0.$$

(Pham Kim Hung)

SOLUTION. We use the global derivative as in the previous solution. Notice that this inequality is obvious due to **Schur** inequality if one of four numbers a, b, c, d is equal to 0. By taking the global derivative of the left-hand side expression, we only need to prove that

$$\sum_{cyc}(a-b)(a-c)(a-d) + \sum_{cyc} abc \geq 0.$$

Using the mixing all variables method, if suffices to prove it in case $\min\{a, b, c, d\} = 0$. WLOG, assume that $a \geq b \geq c \geq d = 0$. The inequality becomes

$$a(a-b)(a-c) + b(b-a)(b-c) + c(c-a)(c-b) \geq 0,$$

which is exactly **Schur** inequality for three numbers;, so we are done. The equality holds for $a = b = c, d = 0$ or permutations.

∇

Example 1.2.3. *Let a, b, c, d be non-negative real numbers such that $a^2 + b^2 + c^2 + d^2 = 4$. Prove that*

$$a(a-b)(a-c)(a-d) + b(b-a)(b-c)(b-d) + c(c-a)(c-b)(c-d)+$$

$$+d(d-a)(d-b)(d-c) \geq abcd - 1.$$

(Pham Kim Hung)

SOLUTION. We have to prove that

$$16 \sum a(a-b)(a-c)(a-d) + (a^2+b^2+c^2+d^2)^2 - 16abcd \geq 0.$$

If $d = 0$, the inequality is obvious due to **AM-GM** inequality and **Schur** inequality (for three numbers). According to the mixing all variables method and the global derivative, it suffices to prove that

$$16 \sum_{cyc} (a-b)(a-c)(a-d) + 4(a+b+c+d)(a^2+b^2+c^2+d^2) - 16 \sum_{cyc} abc \geq 0.$$

or

$$4 \sum_{cyc} (a-b)(a-c)(a-d) + (a+b+c+d)(a^2+b^2+c^2+d^2) - 4 \sum_{cyc} abc \geq 0 \ (\star)$$

If one of four numbers a, b, c, d, say d, is equal to 0, then the previous inequality becomes

$$4 \sum_{cyc}^{a,b,c} a(a-b)(a-c) + (a+b+c)(a^2+b^2+c^2) - 8abc \geq 0,$$

which is obvious due to the following applications of **Schur** inequality and **AM-GM** inequality

$$4 \sum_{cyc}^{a,b,c} a(a-b)(a-c) \geq 0 \ ;$$

$$(a+b+c)(a^2+b^2+c^2) - 8abc \geq 9abc - 8abc \geq 0 \ ;$$

Therefore, according to the mixing all variables method, in order to prove $(\star)$ by taking the global derivative, it suffices to prove that

$$4 \sum_{cyc}^{a,b,c,d} a^2 + 2 \left(\sum_{cyc}^{a,b,c,d} a \right)^2 \geq 8 \sum_{sym} ab \ (\star\star)$$

Clearly, **AM-GM** inequality yields that

$$4 \sum_{cyc}^{a,b,c,d} a^2 \geq \frac{8}{3} \sum_{sym} ab \ ; \ 2 \left(\sum_{cyc}^{a,b,c,d} a \right)^2 \geq \frac{16}{3} \sum_{sym} ab \ ;$$

Adding up the results above, we get $(\star\star)$ and then $(\star)$. The conclusion follows and the equality holds for $a = b = c = d = 1$.

∇

This should satisfy anyone who desperately wanted a **Schur** inequality in 4 variables. What happens for the case $n = 5$? Generalizations are a bit more complicated.

Example 1.2.4. *Let a, b, c, d, e be non-negative real numbers such that $a+b+c+d+e=1$. Prove that*

$$a(a-b)(a-c)(a-d)(a-e)+b(b-a)(b-c)(b-d)(b-e)+c(c-a)(c-b)(c-d)(c-e)+$$
$$+d(d-a)(d-b)(d-c)(d-e)+e(e-a)(e-b)(e-c)(e-d) \geq \frac{-1}{4320}.$$

(Pham Kim Hung)

SOLUTION. To prove this problem, we have to use two of the previous results. Our inequality is equivalent to

$$\frac{1}{4320}(a+b+c+d+e)^5 + \sum_{cyc} a(a-b)(a-c)(a-d)(a-e) \geq 0.$$

Taking the global derivative, we have to prove that

$$\frac{5}{860}(a+b+c+d+e)^4 + \sum_{cyc}(a-b)(a-c)(a-d)(a-e) \geq 0.$$

Due to the mixing all variables method, we only need to check this inequality in case $\min\{a, b, c, d, e\} = 0$. WLOG, assume that $a \geq b \geq c \geq d \geq e = 0$. The inequality becomes

$$\frac{5}{860}(a+b+c+d)^4 + \sum_{cyc}^{a,b,c,d} a(a-b)(a-c)(a-d) \geq 0.$$

This inequality is true according to example 1.2.1 because $\frac{5}{860} \geq \frac{1}{432}$. This shows that it suffices to consider the first inequality in case $a \geq b \geq c \geq d \geq e = 0$. In this case, the inequality becomes

$$\frac{1}{4320}(a+b+c+d)^5 + \sum_{cyc}^{a,b,c,d} a^2(a-b)(a-c)(a-d) \geq 0.$$

If one of the numbers a, b, c, d is equal to 0, the inequality is true by **Schur** inequality so we only need to prove that (by taking the global derivative)

$$\frac{1}{216}(a+b+c+d)^4 + 2\sum_{cyc}^{a,b,c,d} a(a-b)(a-c)(a-d) \geq 0$$

or

$$\frac{1}{432}(a+b+c+d)^4 + \sum_{cyc}^{a,b,c,d} a(a-b)(a-c)(a-d) \geq 0.$$

This inequality is exactly the inequality in example 1.2.2. The proof is completed and we cannot have equality.

∇

Example 1.2.5. *Let a, b, c, d, e be non-negative real numbers. Prove that*

$$a(a-b)(a-c)...(a-e)+b(b-a)(b-c)...(b-e)+c(c-a)(c-b)(c-d)(c-e)+$$

$$+d(d-a)...(d-c)(d-e)+e(e-a)(e-b)...(e-d)+a^2bcd+b^2cde+c^2dea+d^2eab+e^2abc \geq 0.$$

SOLUTION. This problem is easier than the previous problem. Taking the global derivative for a first time, we obtain an obvious inequality

$$\sum_{cyc}(a-b)(a-c)(a-d)(a-e)+2\sum_{cyc}abcd+\sum_{cyc}a^2(bc+cd+da) \geq 0$$

which is true when one of the numbers a, b, c, d, e is equal to 0. Now we only need to prove the initial inequality in case $\min\{a, b, c, d, e\} = 0$. WLOG, assume that $e = 0$, then the inequality becomes

$$\sum_{cyc}^{a,b,c,d} a^2(a-b)(a-c)(a-d)+a^2bcd \geq 0.$$

If one of a, b, c, d is equal to 0, the inequality is true due to **Schur** inequality. Therefore we only need to prove that (taking the global derivative for the second time)

$$2\sum_{cyc}^{a,b,c,d} a(a-b)(a-c)(a-d)+2abcd+a^2(bc+cd+da) \geq 0.$$

This inequality is true according to example 1.2.2 and we are done immediately. The equality holds if and only if three of the five numbers a, b, c, d, e are equal to each other and the two remaining numbers are equal to 0.

∇

These problems have given us a strong expectation of something similar in the general case of n variables. Of course, everything becomes much harder in this case, and we will need to use induction.

Example 1.2.6. *Let $a_1, a_2, ..., a_n$ be non-negative real numbers such that $a_1+a_2+...+a_n = 1$. For $c = -9 \cdot 2^{2n-7}n(n-1)(n-2)$, prove that*

$$a_1(a_1-a_2)...(a_1-a_n)+a_2(a_2-a_3)...(a_2-a_n)+...+a_n(a_n-a_1)(a_n-a_2)...(a_n-a_{n-1}) \geq \frac{1}{c}.$$

(Pham Kim Hung)

SOLUTION. To handle this problem, we need to prove it in the general case, that means, find an estimation of

$$F_{k,n} = \sum_{i=1}^{n}\left(a_i^k \prod_{j=1, j\neq i}^{n}(a_i-a_j)\right),$$

where the non-negative real numbers $a_1, a_2, ..., a_n$ have sum 1. After a process of guessing and checking induction steps, we find out

$$\sum_{i=1}^{n}\left(a_i^k \prod_{j=1, j\neq i}^{n}(a_i - a_j)\right) \geq \frac{3^{-2}2^{9-2n-2k}}{(n+k-1)(n+k-2)(n+k-3)}.$$

Let's construct the following sequence for all $k \geq 1, n \geq 4$

$$c_{k,n} = 9 \cdot 2^{2n+2k-9}(n+k-1)(n+k-2)(n+k-3).$$

We will prove the following general result by induction

$$\frac{1}{c_{k,n}}\left(\sum_{i=1}^{n} a_i\right)^{k+n-1} + \sum_{i=1}^{n}\left(a_i^k \prod_{j=1, j\neq i}^{n}(a_i - a_j)\right) \geq 0 \ (\star)$$

We use induction for $m = k + n$, and we assume that $(\star)$ is already true for all n', k' such that $k' + n' \leq m$. We will prove that $(\star)$ is also true for all n, k such that $n + k = m + 1$. Indeed, after taking the global derivative, the inequality $(\star)$ becomes

$$\frac{n(n+k-1)}{c_{k,n}}\left(\sum_{i=1}^{n} a_i\right)^{k+n-1} + k\sum_{i=1}^{n}\left(a_i^{k-1} \prod_{j=1, j\neq i}^{n}(a_i - a_j)\right) \geq 0 \ (\star\star)$$

According to the inductive hypothesis (for n and $k-1$), we have

$$\frac{1}{c_{k-1,n}}\left(\sum_{i=1}^{n} a_i\right)^{k+n-1} + \sum_{i=1}^{n}\left(a_i^{k-1} \prod_{j=1, j\neq i}^{n}(a_i - a_j)\right) \geq 0$$

Moreover, because

$$\frac{n(n+k-1)}{kc_{k,n}} \geq \frac{1}{c_{k-1,n}} \ \forall n \geq 4,$$

the inequality $(\star\star)$ is successfully proved. By the mixing all variables method, we only need to consider $(\star)$ in case $\min\{a_1, a_2, ..., a_n\} = 0$. WLOG, assume that $a_1 \geq a_2 \geq ... \geq a_n$, then $a_n = 0$ and the inequality becomes

$$\frac{1}{c_{k,n}}\left(\sum_{i=1}^{n-1} a_i\right)^{k+n-1} + \sum_{i=1}^{n-1}\left(a_i^{k+1} \prod_{j=1, j\neq i}^{n-1}(a_i - a_j)\right) \geq 0$$

or (since $c_{k,n} = c_{k+1,n-1}$)

$$\frac{1}{c_{k+1,n-1}}\left(\sum_{i=1}^{n-1} a_i\right)^{k+n-1} + \sum_{i=1}^{n-1}\left(a_i^{k+1} \prod_{j=1, j\neq i}^{n-1}(a_i - a_j)\right) \geq 0.$$

This is another form of the inequality $(\star)$ for $n-1$ numbers $a_1, a_2, ..., a_{n-1}$ and for the exponent $k+1$ (instead of k). Performing this reasoning $n-4$ times (or we can use induction again), we can change $(\star)$ to the problem of only four numbers a_1, a_2, a_3, a_4 but with the exponents $k+n-4$. Namely, we have to prove that

$$(a+b+c+d)^{k+n-1} + M\sum_{cyc} a^{k+n-4}(a-b)(a-c)(a-d) \geq 0 \quad (\star\star\star)$$

where $M = c_{k+n-4,4} = c_{k,n}$. Taking the global derivative of $(\star\star\star)$ exactly r times $(r \leq k+n-4)$, we obtain the following inequality

$$4^r(k+n-1)(k+n-2)...(k+n-r)(a+b+c+d)^4 +$$
$$+(k+n-4)(k+n-3)...(k+n-r-3)M\sum_{cyc} a^{k+n-4-r}(a-b)(a-c)(a-d) \geq 0 \quad [\star]$$

We will call the inequality constructed by taking r times the global derivative of $(\star)$ as the $[r^{th}]$ inequality (this previous inequality is the $[r^{th}]$ inequality). If $abcd = 0$, assume $d = 0$, and the $[r^{th}]$ inequality is true for all $r \in \{0, 1, 2, ..., n+k-5\}$ because

$$\sum_{cyc} a^{k+n-4-r}(a-b)(a-c)(a-d) = \sum_{cyc} a^{k+n-3-r}(a-b)(a-c) \geq 0.$$

According to the principles of the mixing all variables method and global derivative, if the $[(r+1)^{th}]$ inequality is true for all a, b, c, d and the $[r^{th}]$ inequality is true when $abcd = 0$ then the $[r^{th}]$ inequality is true for all non-negative real numbers a, b, c, d. Because $abcd = 0$, the $[r^{th}]$ is true for all $0 \leq r \leq n+k-5$, so we conclude that, in order to prove the $[0^{th}]$ inequality (which is exactly $(\star\star\star)$), we only need to check the $[(n+k-4)^{th}]$ inequality. The $[(n+k-4)^{th}]$ inequality is as follows

$$4^{n+k-4}(k+n-1)(k+n-2)...(4)(a+b+c+d)^4 +$$
$$+(k+n-4)(k+n-3)...(1)M\sum_{cyc}(a-b)(a-c)(a-d) \geq 0$$
$$\Leftrightarrow\ 4^{n+k-4}(k+n-1)(k+n-2)(k+n-3)(a+b+c+d)^4 + 6M\sum_{cyc}(a-b)(a-c)(a-d) \geq 0$$
$$\Leftrightarrow\ (a+b+c+d)^4 + 27\sum_{cyc}(a-b)(a-c)(a-d) \geq 0.$$

This last inequality is clearly true by the mixing all variables method and **AM-GM** inequality. The inductive process is finished and the conclusion follows immediately.

★ *Consider the non-negative real numbers $a_1, a_2, ..., a_n$ such that $a_1 + a_2 + ... + a_n = 1$. For $k = 9 \cdot 2^{2n+2k-9}(n+k-1)(n+k-2)(n+k-3)$, we have*

$$\sum_{i=1}^{n}\left(a_i^k \prod_{j=1, j\neq i}^{n}(a_i - a_j)\right) \geq \frac{-1}{k}.$$

∇

Article 2

Looking at Familiar Expressions

1.3 On AM-GM Inequality

Certainly, if a, b, c are positive real numbers then **AM-GM** inequality shows that

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq 3\sqrt[3]{\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a}} = 3.$$

We denote $G(a, b, c) = \frac{a}{b} + \frac{b}{c} + \frac{c}{a} - 3$, then $G(a, b, c) \geq 0$ for all $a, b, c > 0$. This article will present some nice properties regarding the function G.

Example 1.3.1. *Let a, b, c, k be positive real numbers. Prove that*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a+k}{b+k} + \frac{b+k}{c+k} + \frac{c+k}{a+k}.$$

SOLUTION. Notice that we can transform the expression $G(a, b, c)$ into

$$G(a, b, c) = \left(\frac{a}{b} + \frac{b}{a} - 2\right) + \left(\frac{b}{c} + \frac{c}{a} - \frac{b}{a} - 1\right) = \frac{(a-b)^2}{ab} + \frac{(a-c)(b-c)}{ac}.$$

WLOG, assume that $c = \min(a, b, c)$. Our inequality is equivalent to

$$\frac{(a-b)^2}{ab} + \frac{(a-c)(b-c)}{ac} \geq \frac{(a-b)^2}{(a+k)(b+k)} + \frac{(a-c)(b-c)}{(a+k)(c+k)}.$$

Because $c = \min(a, b, c)$ it follows that $(a-c)(b-c) \geq 0$ and the inequality is obvious. The proof is completed and equality holds for $a = b = c$.

∇

Example 1.3.2. *Let a, b, c be positive real numbers. If $k \geq \max(a^2, b^2, c^2)$, prove that*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \frac{a^2+k}{b^2+k} + \frac{b^2+k}{c^2+k} + \frac{c^2+k}{a^2+k}.$$

(Pham Kim Hung)

SOLUTION. Similarly as in the preceding inequality, this one is equivalent to

$$\frac{(a-b)^2}{ab}+\frac{(a-c)(b-c)}{ac}\geq\frac{(a-b)^2(a+b)^2}{(a^2+k)(b^2+k)}+\frac{(a-c)(b-c)(a+c)(b+c)}{(a^2+k)(c^2+k)}.$$

WLOG, assume that $c=\min(a,b,c)$. It's sufficient to prove that

$$(a^2+k)(b^2+k)\geq ab(a+b)^2 \ ;$$

$$(a^2+k)(c^2+k)\geq ac(a+c)(b+c) \ .$$

The first one is certainly true because

$$(a^2+k)(b^2+k)\geq (a^2+b^2)^2\geq ab(a+b)^2.$$

The second one is equivalent to

$$c^2(k-ac)+a^2(k-bc)+k^2-abc^2\geq 0$$

which is also obvious because $k\geq\max(a^2,b^2,c^2)$. We are done.

Comment. The following inequality is stronger

★ *Let a,b,c be positive real numbers. If $k\geq\max(ab,bc,ca)$, prove that*

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a^2+k}{b^2+k}+\frac{b^2+k}{c^2+k}+\frac{c^2+k}{a^2+k}.$$

To prove this one, we only note that if $c=\min(a,b,c)$ and $k\geq\max(ab,bc,ca)$ then

$$(a^2+k)(b^2+k)\geq(a^2+ab)(b^2+ab)=ab(a+b)^2 \ ;$$

$$c^2(k-ac)+a^2(k-bc)+k^2-abc^2\geq c^2(k-ac)+a^2(k-bc)+(ac)\cdot(bc)-abc^2=0 \ .$$

The equality holds for $a=b\leq c$ and $k=ac$. Both a and c can take arbitrary values.

∇

Example 1.3.3. *If a,b,c are the side lengths of a triangle, then*

$$4\left(\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\right)\geq 9+\frac{a^2+c^2}{c^2+b^2}+\frac{c^2+b^2}{b^2+a^2}+\frac{b^2+a^2}{a^2+c^2}.$$

(Pham Kim Hung)

SOLUTION. The inequality can be rewritten in the following form

$$\frac{4(a-b)^2}{ab}+\frac{4(c-a)(c-b)}{ac}\geq\frac{(a^2-b^2)^2}{(a^2+c^2)(c^2+b^2)}+\frac{(c^2-a^2)(c^2-b^2)}{(a^2+b^2)(a^2+c^2)}$$

WLOG, we may assume $c = \min(a, b, c)$. Then it's not too difficult to show that

$$\frac{1}{ac} \geq \frac{(c+a)(c+b)}{(a^2+b^2)(c^2+b^2)}$$

$$\frac{4}{ab} \geq \frac{(a+b)^2}{(a^2+c^2)(b^2+c^2)}$$

and the proof is completed. Equality holds for $a = b = c$.

∇

Example 1.3.4. *Let a, b, c be positive real numbers. Prove that*

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} + \frac{8(ab+bc+ca)}{a^2+b^2+c^2} \geq 11.$$

(Nguyen Van Thach)

SOLUTION. Similarly, this inequality can be rewritten in the following form

$$(a-b)^2\left(\frac{(a+b)^2}{a^2b^2} - \frac{8}{a^2+b^2+c^2}\right) + (c-a)(c-b)\left(\frac{(a+c)(b+c)}{a^2c^2} - \frac{8}{a^2+b^2+c^2}\right) \geq 0.$$

WLOG, assume that $c = \min\{a, b, c\}$. We have

$$\frac{(a+b)^2}{a^2b^2} \geq \frac{8}{a^2+b^2} \geq \frac{8}{a^2+b^2+c^2}.$$

Moreover,

$$(a+c)(b+c)(a^2+b^2+c^2) \geq 2c(a+c)(a^2+2c^2) \geq 8a^2c^2$$

$$\Rightarrow \frac{(a+c)(b+c)}{a^2c^2} \geq \frac{8}{a^2+b^2+c^2}.$$

Therefore we get the desired result. Equality holds for $a = b = c$.

∇

Example 1.3.5. *Let a, b, c be the side lengths of a triangle. Prove that*

$$\frac{a^2+b^2}{a^2+c^2} + \frac{c^2+a^2}{c^2+b^2} + \frac{b^2+c^2}{b^2+a^2} \geq \frac{a+b}{a+c} + \frac{a+c}{b+c} + \frac{b+c}{b+a}.$$

(Vo Quoc Ba Can)

SOLUTION. It is easy to rewrite the inequality in the following form

$$(a-b)^2M + (c-a)(c-b)N \geq 0,$$

where

$$M = \frac{(a+b)^2}{(a^2+c^2)(b^2+c^2)} - \frac{1}{(a+c)(b+c)} \ ;$$

$$N = \frac{(a+c)(b+c)}{(a^2+b^2)(a^2+c^2)} - \frac{1}{(a+c)(a+b)} \ .$$

WLOG, assume that $c = \min\{a,b,c\}$. Clearly, $M \geq 0$ and $N \geq 0$ since

$$(a+c)^2(b+c)(a+b) - (a^2+b^2)(a^2+c^2) = a^3(b+c-a) + 3a^2bc + 3abc^2 + a^2c^2 + c^3(a+b) \geq 0.$$

We are done. Equality holds for $a = b = c$.

∇

Example 1.3.6. *For all distinct real numbers a, b, c, prove that*

$$\frac{(a-b)^2}{(b-c)^2} + \frac{(b-c)^2}{(c-a)^2} + \frac{(c-a)^2}{(a-b)^2} \geq 5.$$

(Darij Grinberg)

SOLUTION. This inequality is directly deduced from the following identity

$$\frac{(a-b)^2}{(b-c)^2} + \frac{(b-c)^2}{(c-a)^2} + \frac{(c-a)^2}{(a-b)^2} = 5 + \left(1 + \frac{a-b}{b-c} + \frac{b-c}{c-a} + \frac{c-a}{a-b}\right)^2.$$

Comment. According to this identity, we can obtain the following results

★ *Let a, b, c be distinct real numbers. Prove that*

$$G\left((a-b)^2, (b-c)^2, (c-a)^2\right) \geq \left(8 + 3\sqrt{8}\right) G(a-b, b-c, c-a).$$

$$G\left((a-b)^2, (b-c)^2, (c-a)^2\right) + G\left((c-a)^2, (b-c)^2, (a-b)^2\right) \geq \frac{9}{2}.$$

∇

Example 1.3.7. *Prove that for all positive real numbers a, b, c, we have*

$$\frac{a}{b} + \frac{b}{c} + \frac{c}{a} \geq \sqrt{\frac{a^2+c^2}{b^2+c^2}} + \sqrt{\frac{c^2+b^2}{a^2+b^2}} + \sqrt{\frac{b^2+a^2}{c^2+a^2}}$$

SOLUTION. **First Solution.** First, we will prove that

$$\sum_{cyc} \frac{a}{b} + \sum_{cyc} \frac{b}{a} \geq \sum_{cyc} \sqrt{\frac{a^2+c^2}{b^2+c^2}} + \sum_{cyc} \sqrt{\frac{b^2+c^2}{a^2+c^2}} \tag{1}$$

In order to prove (1), we only need to prove that

$$\frac{a}{b} + \frac{b}{a} \geq \sqrt{\frac{a^2+c^2}{b^2+c^2}} + \sqrt{\frac{b^2+c^2}{a^2+c^2}}$$

Indeed, by squaring, this inequality becomes

$$\frac{a^2+b^2}{ab} \geq \frac{a^2+b^2+2c^2}{\sqrt{(a^2+c^2)(b^2+c^2)}}$$

or

$$\left(a^2+b^2\right)\sqrt{(a^2+c^2)(b^2+c^2)} \geq ab\left(a^2+b^2+2c^2\right)$$

or

$$\left(a^2+b^2\right)^2(a^2+c^2)(b^2+c^2) \geq a^2b^2\left(a^2+b^2+2c^2\right)^2$$

or

$$c^2(a^2-b^2)^2\left(a^2+b^2+c^2\right) \geq 0,$$

which is clearly true. As a result, (1) is proved. Now, returning to our problem, we assume by contradiction that the inequality

$$\sum_{cyc}\frac{a}{b} < \sum_{cyc}\sqrt{\frac{a^2+c^2}{b^2+c^2}} \tag{2}$$

is false for a certain triple (a, b, c). By (1), we have

$$\sum_{cyc}\frac{a}{b} + \sum_{cyc}\frac{b}{a} \geq \sum_{cyc}\sqrt{\frac{a^2+c^2}{b^2+c^2}} + \sum_{cyc}\sqrt{\frac{b^2+c^2}{a^2+c^2}}$$

Combining this with (2), we get that

$$\sum_{cyc}\frac{b}{a} > \sum_{cyc}\sqrt{\frac{b^2+c^2}{a^2+c^2}} \tag{3}$$

On the other hand, from (2), we have that

$$\left(\sum_{cyc}\frac{a}{b}\right)^2 < \left(\sum_{cyc}\sqrt{\frac{a^2+c^2}{b^2+c^2}}\right)^2$$

$$\Leftrightarrow \sum_{cyc}\frac{a^2}{b^2} + 2\sum_{cyc}\frac{b}{a} < \sum_{cyc}\frac{a^2+c^2}{b^2+c^2} + 2\sum_{cyc}\sqrt{\frac{b^2+c^2}{a^2+c^2}}.$$

Combining with (3), we obtain

$$\sum_{cyc}\frac{a^2}{b^2} < \sum_{cyc}\frac{a^2+c^2}{b^2+c^2} \tag{4}$$

This inequality contradicts the result in example 1.3.1. Therefore, the assumption is false, or in other words, the inequality is proved successfully. Equality holds for

$a = b = c$.

Second Solution. Recall the following result, presented in the first volume of this book.

Let $a_1, a_2, .., a_n$ and $b_1 \geq b_2 \geq ... \geq b_n \geq 0$ be positive real numbers such that $a_1 a_2 ... a_k \geq b_1 b_2 ... b_k \; \forall k \in \{1, 2, ..., n\}$, then $a_1 + a_2 + ... + a_n \geq b_1 + b_2 + ... + b_n$.

For the case $n = 3$, we obtain the following result

Given positive numbers a, b, c, x, y, z such that $\max\{a, b, c\} \geq \max\{x, y, z\}$, $\min\{a, b, c\} \leq \min\{x, y, z\}$, then $a + b + c \geq x + y + z$.

According to this result, we will prove a general inequality for all $a, b, c, x > 0$ as follows

$$\sum_{cyc} \frac{a}{b} \geq \sum_{cyc} \left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}}.$$

Indeed, we already have

$$\frac{a}{b} \cdot \frac{b}{c} \cdot \frac{c}{a} = \left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}} \cdot \left(\frac{c^x + b^x}{a^x + b^x} \right)^{\frac{1}{x}} \cdot \left(\frac{b^x + a^x}{c^x + a^x} \right)^{\frac{1}{x}} = 1.$$

It suffices to show that

$$\max\left\{\frac{a}{b}, \frac{b}{c}, \frac{c}{a}\right\} \geq \max\left\{ \left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}}, \left(\frac{c^x + b^x}{a^x + b^x} \right)^{\frac{1}{x}}, \left(\frac{b^x + a^x}{c^x + a^x} \right)^{\frac{1}{x}} \right\} \tag{5}$$

$$\min\left\{\frac{a}{b}, \frac{b}{c}, \frac{c}{a}\right\} \leq \min\left\{ \left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}}, \left(\frac{c^x + b^x}{a^x + b^x} \right)^{\frac{1}{x}}, \left(\frac{b^x + a^x}{c^x + a^x} \right)^{\frac{1}{x}} \right\} \tag{6}$$

We prove (5) (and (6) can be proved similarly). WLOG, assume that

$$\left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}} = \max\left\{ \left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}}, \left(\frac{c^x + b^x}{a^x + b^x} \right)^{\frac{1}{x}}, \left(\frac{b^x + a^x}{c^x + a^x} \right)^{\frac{1}{x}} \right\}.$$

We see that $\left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}} \geq 1$, gives $a \geq b$. Therefore

$$\frac{a^x}{b^x} \geq \frac{a^x + c^x}{b^x + c^x} \;\Rightarrow\; \frac{a}{b} \geq \left(\frac{a^x + c^x}{b^x + c^x} \right)^{\frac{1}{x}}.$$

This ends the proof. Equality holds for $a = b = c$.

∇

Example 1.3.8. *Let a, b, c be distinct real numbers. Prove that*

$$\frac{(a-b)^2}{(b-c)^2} + \frac{(b-c)^2}{(c-a)^2} + \frac{(c-a)^2}{(a-b)^2} \geq \frac{a+b}{b+c} + \frac{b+c}{c+a} + \frac{c+a}{a+b}.$$

(Pham Kim Hung)

SOLUTION. WLOG, we may assume that $c = \min(a, b, c)$. Taking into account the preceding example 1.3.1, we deduce that

$$G(a+b, b+c, c+a) \geq G(a+b-2c, b+c-2c, c+a-2c).$$

Let now $x = a - c, y = b - c$, then it remains to prove that (after we consider $c = 0$)

$$\frac{(x-y)^2}{y^2} + \frac{y^2}{x^2} + \frac{x^2}{(x-y)^2} \geq \frac{x+y}{y} + \frac{y}{x} + \frac{x}{x+y}$$

$$\Leftrightarrow \frac{x^2}{y^2} + \frac{y^2}{x^2} + \frac{x^2}{(x-y)^2} \geq \frac{3x}{y} + \frac{y}{x} + \frac{x}{x+y}.$$

From here, we need to consider some smaller cases

(i). The first case. If $x \geq y$ then

Case $y \leq x \leq 2y$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + \frac{9}{4} \geq \frac{3x}{y},\ \frac{y^2}{x^2} + \frac{1}{4} \geq \frac{y}{x},\ \frac{x^2}{(x-y)^2} \geq 4,\ \frac{2}{3} \geq \frac{x}{x+y}.$$

Case $2y \leq x \leq 2.3y$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + 2 \geq \frac{3x}{y},\ \frac{y^2}{x^2} + \frac{1}{4} \geq \frac{y}{x},\ \frac{x^2}{(x-y)^2} \geq 3.1,\ \frac{2.3}{3.3} \geq \frac{x}{x+y}.$$

Case $2.3y \leq x \leq 2.5y$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + 1.61 \geq \frac{3x}{y},\ \frac{y^2}{x^2} + \frac{1}{4} \geq \frac{y}{x},\ \frac{x^2}{(x-y)^2} \geq 2.77,\ \frac{2.5}{3.5} \geq \frac{x}{x+y}.$$

Case $2.5y \leq x \leq 3y$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + 1.25 \geq \frac{3x}{y},\ \frac{y^2}{x^2} + \frac{1}{4} \geq \frac{y}{x},\ \frac{x^2}{(x-y)^2} \geq 2.25,\ \frac{3}{4} \geq \frac{x}{x+y}.$$

Case $3y \leq x \leq 4y$. The desired result is obtained by adding

$$\frac{x^2}{y^2} \geq \frac{3x}{y},\ \frac{y^2}{x^2} + \frac{1}{4} \geq \frac{y}{x},\ \frac{x^2}{(x-y)^2} \geq 1.77,\ \frac{4}{5} \geq \frac{x}{x+y}.$$

Case $x \geq 4y$. The desired result is obtained by adding

$$\frac{x^2}{y^2} \geq \frac{3x}{y} + 4,\ \frac{y^2}{x^2} + \frac{1}{4} \geq \frac{y}{x},\ \frac{x^2}{(x-y)^2} \geq 1,\ 1 \geq \frac{x}{x+y}.$$

(ii). The second case. If $x \leq y$ then

Case $x \le y \le 1.5x$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + 2 \ge \frac{3x}{y},\ \frac{y^2}{x^2} \ge \frac{y}{x},\ \frac{x^2}{(y-x)^2} \ge 4,\ \frac{1}{2} \ge \frac{x}{x+y}.$$

Case $1.5x \le y \le 1.8x$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + \frac{14}{9} \ge \frac{3x}{y},\ \frac{y^2}{x^2} \ge \frac{y}{x} + 0.75,\ \frac{x^2}{(y-x)^2} \ge 1.56,\ \frac{1}{2.5} \ge \frac{x}{x+y}.$$

Case $1.8x \le y \le 3x$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + 1.4 \ge \frac{3x}{y},\ \frac{y^2}{x^2} \ge \frac{y}{x} + 1.44,\ \frac{x^2}{(y-x)^2} \ge \frac{1}{4},\ \frac{1}{2.8} \ge \frac{x}{x+y}.$$

Case $y \ge 3x$. The desired result is obtained by adding

$$\frac{x^2}{y^2} + \frac{8}{9} \ge \frac{3x}{y},\ \frac{y^2}{x^2} \ge \frac{y}{x} + 6,\ \frac{x^2}{(y-x)^2} \ge 0,\ 1 \ge \frac{x}{x+y}.$$

The proof has been proved completely. There's no equality case.

∇

1.4 On Nesbitt's Inequality

The famous **Nesbitt** inequality has the following form

★ *If a, b, c are positive real numbers then*

$$N(a,b,c) = \frac{a}{b+c} + \frac{b}{c+a} + \frac{c}{a+b} - \frac{3}{2} \geq 0.$$

In the following pages, we will discuss some inequalities which have the same appearance as **Nesbitt** inequality and we will also discuss some nice properties of $N(a,b,c)$. First we give a famous generalization of **Nesbitt** inequality with real exponents.

Example 1.4.1. *Let a, b, c be non-negative real numbers. For each real number k, find the minimum of the following expression*

$$S = \left(\frac{a}{b+c}\right)^k + \left(\frac{b}{c+a}\right)^k + \left(\frac{c}{a+b}\right)^k.$$

SOLUTION. Certainly, **Nesbitt** inequality is a particular case of this inequality for $k=1$. If $k \geq 1$ or $k \leq 0$ then it's easy to deduce that

$$\left(\frac{a}{b+c}\right)^k + \left(\frac{b}{c+a}\right)^k + \left(\frac{c}{a+b}\right)^k \geq \frac{3}{2^k}.$$

If $k = \frac{1}{2}$, we obtain a familiar result as follows

$$\sqrt{\frac{a}{b+c}} + \sqrt{\frac{b}{a+c}} + \sqrt{\frac{c}{a+b}} \geq 2.$$

The most difficult case is $0 < k < 1$. We will prove by mixing variables that

$$f(a,b,c) = \left(\frac{a}{b+c}\right)^k + \left(\frac{b}{c+a}\right)^k + \left(\frac{c}{a+b}\right)^k \geq \min\left\{2, \frac{3}{2^k}\right\}.$$

WLOG, we may assume that $a \geq b \geq c$. Let $t = \frac{a+b}{2} \geq u = \frac{a-b}{2} \geq 0$ then

$$\sum_{cyc} \left(\frac{a}{b+c}\right)^k = g(u) = \left(\frac{u+t}{t-u+c}\right)^k + \left(\frac{t-u}{u+t+c}\right)^k + \left(\frac{c}{2t}\right)^k.$$

We infer that the derivative

$$g'(u) = k\left(\frac{u+t}{u-t+c}\right)^{k-1} \frac{2t+c}{(u-t+c)^2} + k\left(\frac{t-u}{u+t-c}\right)^{k-1} \frac{-2t-c}{(u+t+c)^2}$$

has the same sign as the following function

$$h(u) = (k-1)\big[\ln(t+u) - \ln(t-u)\big] + (k+1)\big[\ln(u+t+c) - \ln(t-u+c)\big].$$

It's easy to check that

$$h'(u) = \frac{2t(k-1)}{t^2-u^2} + \frac{2(k+1)(t+c)}{(t+c)^2-u^2}.$$

Because $t \geq c$ it follows that $t(t+c)(t-c) \geq (t+2c)u^2$ and therefore $2(t+c)(t^2-u^2) \geq t\left((t+c)^2-u^2\right)$. If $k+1 > 2(1-k)$ (or $k > 1/3$), we have indeed that $h'(u) > 0$. Thus $h(u) \geq h(0) = 0 \;\Rightarrow g'(u) \geq 0$. Therefore g is a monotonic function if $u \geq 0$ and therefore

$$\sum_{cyc}\left(\frac{a}{b+c}\right)^k \geq g(u) \geq g(0) = \frac{2t^k}{(t+c)^k} + \frac{c^k}{(2t)^k} \;(\star)$$

Because $g(0)$ is homogeneous, we may assume that $c \leq t = 1$. Consider the function

$$p(c) = \frac{2^{k+1}}{(1+c)^k} + c^k$$

Since the derivative

$$p'(c) = \frac{-k.2^{k+1}}{(1+c)^{k+1}} + kc^{k-1}$$

has the same sign as the function

$$q(c) = (k+1)\ln(c+1) + (k-1)\ln c - (k+1)\ln 2$$

and also because, as it's easy to check

$$q'(c) = \frac{k+1}{c+1} + \frac{k-1}{c} = \frac{(k+1)c + (c+1)(k-1)}{c(c+1)}$$

has no more than one real root, we obtain that $p'(c) = 0$ has no more than one real root in $(0,1)$ (because $p'(1) = 0$). Furthermore, $\lim\limits_{c\to 0} p(c) = +\infty$, so we obtain

$$p(c) \geq \min\left\{p(1)\;;\;\lim_{c\to 0} p(c)\right\} = \min\left\{3\;;\;2^{k+1}\right\} \;(\star\star)$$

According to $(\star)$ and $(\star\star)$, we conclude that

$$\left(\frac{a}{b+c}\right)^k + \left(\frac{b}{a+c}\right)^k + \left(\frac{c}{a+b}\right)^k \geq \min\left\{\frac{3}{2^k}\;;\;2\right\}.$$

The inequality has been proved in case $k > 1/3$, now we will consider the case $k \leq 1/2$. Choose three numbers α, β, γ satisfying that $\sqrt{\alpha} = a^k, \sqrt{\beta} = b^k, \sqrt{\gamma} = c^k$, then

$$(b+c)^{2k} \leq b^{2k} + c^{2k} \;\Rightarrow \left(\frac{b}{b+c}\right)^{2k} + \left(\frac{c}{c+b}\right)^{2k} \geq 1 \;\Rightarrow \frac{a^k}{(b+c)^k} \geq \sqrt{\frac{\alpha}{\beta+\gamma}}$$

Constructing similar results and summing up, we get

$$\frac{a^k}{(b+c)^k} + \frac{b^k}{(c+a)^k} + \frac{c^k}{(a+b)^k} \geq \sqrt{\frac{\alpha}{\beta+\gamma}} + \sqrt{\frac{\beta}{\alpha+\gamma}} + \sqrt{\frac{\gamma}{\alpha+\beta}} \geq 2.$$

Therefore the problem has been completely solved, with the conclusion that

$$\left(\frac{a}{b+c}\right)^k+\left(\frac{b}{a+c}\right)^k+\left(\frac{c}{a+b}\right)^k\ge \min\left\{\frac{3}{2^k}\ ;\ 2\right\}.$$

If $k=\dfrac{\ln 3}{\ln 2}-1$, the equality holds for $a=b=c$ and $a=b, c=0$ up to permutation. Otherwise, the equality only holds in the case $a=b=c$.

∇

In volume I we have a problem from the Vietnam TST 2006, where we have already proved that if a,b,c are the side-lengths of a triangle then

$$(a+b+c)\left(\frac{1}{a}+\frac{a}{b}+\frac{1}{c}\right)\ge 6\left(\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}\right),$$

or, in other words, $N(a,b,c)\ge 3N(a+b,b+c,c+a)$. Moreover, we also have some other nice results related to the expression N as follows

Example 1.4.2. *Let a,b,c be the side lengths of a triangle. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{3}{2}\le\frac{2ab}{c(a+b)}+\frac{2bc}{a(b+c)}+\frac{2ca}{b(c+a)}.$$

or, in other words, prove that $N(a,b,c)\le 2N\left(\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}\right)=2N(ab,bc,ca).$

(Pham Kim Hung)

SOLUTION. First, we change the inequality to **SOS** form as follows

$$\sum_{cyc}(2c^2-ab)(a+b)(a-b)^2\ge 0.$$

WLOG, assume that $a\ge b\ge c$, then $S_a\ge S_b\ge S_c$. Therefore, it's enough to prove that

$$b^2S_b+c^2S_c\ge 0\ \Leftrightarrow\ b^2(2b^2-ac)(a+c)+c^2(2c^2-ab)(a+b)\ge 0.$$

This last inequality is obviously true because $b(a+c)\ge c(a+b)$ and $b(2b^2-ac)\ge c(2c^2-ab)$. The equality holds for $a=b=c$ or $(a,b,c)\sim(2,1,1)$.

∇

Example 1.4.3. *Let a,b,c be positive real numbers. Prove that*

$$\frac{2ab}{c(a+b)}+\frac{2bc}{a(b+c)}+\frac{2ca}{b(c+a)}\ge\frac{a+b}{2c+a+b}+\frac{b+c}{2a+b+c}+\frac{c+a}{2b+c+a},$$

or, in other words, prove that $N\left(\dfrac{1}{a},\dfrac{1}{b},\dfrac{1}{c}\right)\ge N(a+b,b+c,c+a).$

SOLUTION. Similarly to the preceding problem, after changing the inequality to **SOS** form, we only need to prove that

$$\frac{a(b^3+c^3)+bc(b^2+c^2)}{abc\prod\limits_{cyc}(a+b)} \geq \frac{2a+3b+3c}{\prod\limits_{cyc}(2a+b+c)}$$

if $a, b, c \geq 0$ and $a \geq b \geq c$. Notice that $b^3+c^3 \geq bc(b+c)$, so

$$\text{LHS} \geq \frac{1}{(a+b)(a+c)}$$

and it remains to prove that

$$\frac{(a+2b+c)(a+b+2c)}{(a+b)(a+c)} \geq \frac{2a+3b+3c}{2a+b+c}$$

$$\Leftrightarrow \frac{2(b+c)^2+2a(b+c)}{(a+b)(a+c)} \geq \frac{2(b+c)}{2a+b+c} \Leftrightarrow \frac{a+b+c}{(a+b)(a+c)} \geq \frac{1}{2a+b+c}.$$

This last condition is obviously true. The equality holds for $a=b=c$.

∇

1.5 On Schur Inequality

Consider the following expression in three variables a, b and c

$$F(a,b,c) = a^3 + b^3 + c^3 + 3abc - ab(a+b) - bc(b+c) - ca(c+a).$$

By the third degree-**Schur** inequality, we have $F(a,b,c) \geq 0$ for all non-negative a, b, c. In this article, we will discover some interesting relations between Schur-like expressions such as $F(a^2, b^2, c^2)$, $F(a+b, b+c, c+a)$, $F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right)$ and $F(a-b, b-c, c-a)$, etc. First, we have

Example 1.5.1. *Let a, b, c be the side lengths of a triangle. Prove that*

$$F(a,b,c) \leq F(a+b, b+c, c+a).$$

(Pham Kim Hung)

SOLUTION. Notice that the expression $F(a,b,c)$ can be rewritten as

$$F(a,b,c) = \sum_{cyc} a(a-b)(a-c).$$

Therefore our inequality is equivalent to

$$\sum_{cyc} (b+c-a)(a-b)(a-b) \geq 0$$

By hypothesis we have $b+c-a \geq 0, c+a-b \geq 0, a+b-c \geq 0$, so the above result follows from the generalized **Schur** inequality. We are done and the equality holds for $a = b = c$ (equilateral triangle) or $a = 2b = 2c$ up to permutation (degenerated triangle).

∇

Example 1.5.2. *Let a, b, c be the side lengths of a triangle. Prove that*

$$F(a,b,c) \leq 4a^2b^2c^2 F\left(\frac{1}{a}, \frac{1}{b}, \frac{1}{c}\right).$$

(Pham Kim Hung)

SOLUTION. Generally, the expression $F(a,b,c)$ can be represented in **SOS** form as

$$F(a,b,c) = \frac{1}{2}\sum_{cyc} (a+b-c)(a-b)^2.$$

Therefore we can change our inequality to the following

$$\sum_{cyc} \left(2c^2\left(\frac{1}{a} + \frac{1}{b} - \frac{1}{c}\right) - (a+b-c)\right)(a-b)^2 \geq 0,$$

and therefore the coefficients S_a, S_b, S_c can be determined from

$$S_c = 2c^2\left(\frac{1}{a}+\frac{1}{b}-\frac{1}{c}\right)-(a+b-c)\ ;$$

$$S_b = 2b^2\left(\frac{1}{a}+\frac{1}{c}-\frac{1}{b}\right)-(c+a-b)\ ;$$

$$S_a = 2a^2\left(\frac{1}{b}+\frac{1}{c}-\frac{1}{a}\right)-(b+c-a)\ .$$

WLOG, assume that $a \ge b \ge c$. Clearly, $S_a \ge 0$ and $S_a \ge S_b \ge S_c$. It suffices to prove that $S_b + S_c \ge 0$ or namely

$$\frac{4(b^2+c^2)}{a}+2\left(\frac{b^2}{c}+\frac{c^2}{b}-b-c\right)-2a \ge 0.$$

Notice that $a \le b+c$, so $2(b^2+c^2) \ge (b+c)^2 \ge a^2$ and we are done because

$$\frac{4(b^2+c^2)}{a} \ge 2a\ ;\ \frac{b^2}{c}+\frac{c^2}{b} \ge b+c\ .$$

The equality holds for $a=b=c$ and $a=2b=2c$ up to permutation.

∇

Example 1.5.3. *Suppose that a, b, c are the side lengths of a triangle. Prove that*

$$F(a^2,b^2,c^2) \le 36F(ab,bc,ca).$$

(Pham Kim Hung)

SOLUTION. Similarly, this inequality can be transformed into

$$\sum_{cyc}(a^2+b^2-c^2)(a^2-b^2)^2 \le 36\sum_{cyc}c^2(ac+bc-ab)(a-b)^2$$

or $S_a(b-c)^2+S_b(c-a)^2+S_c(a-b)^2 \ge 0$, where

$$S_a = 36a^2(ab+ac-bc)-(b+c)^2(b^2+c^2-a^2)\ ;$$

$$S_b = 36b^2(bc+ba-ca)-(c+a)^2(c^2+a^2-b^2)\ ;$$

$$S_c = 36c^2(ca+cb-ab)-(a+b)^2(a^2+b^2-c^2)\ .$$

WLOG, assume that $a \ge b \ge c$, then certainly $S_a \le S_b \le S_c$ and $S_c \ge 0$. Thererfore it suffices to prove that $S_a + S_b \ge 0$, which can be reduced to

$$36bc(b^2+c^2)+34a(b-c)^2(b+c)-(b^2-c^2)^2-a^2(a+b+c)^2-a^4+2a^2bc \ge 0.$$

Suppose that S is the left expression in the previous inequality. Consider the cases *(i). The first case.* If $17(b-c)^2(b+c) \geq a(a+b+c)(2a+b+c)$. Certainly, we have

$$\begin{aligned} S &\geq 34a(b-c)^2(b+c) - 2a^2(a+b+c)^2 - a^4 - (b^2-c^2)^2 \\ &\geq 2a^2(a+b+c)(2a+b+c) - 2a^2(a+b+c)^2 - a^4 - b^4 \\ &= 2a^3(a+b+c) - a^4 - b^4 \geq 0. \end{aligned}$$

(ii). The second case. If $17(b-c)^2(b+c) \leq a(a+b+c)(2a+b+c)$, we get that $a \geq a_0$ where a_0 is the unique real root of the equation

$$17(b-c)^2(b+c) = x(x+b+c)(2x+b+c)).$$

Clearly, $S = S(a)$ is a decreasing function of a, if $a \geq \max(a_0, b)$ (because $S'(a) \leq 0$), so we obtain

$$S = S(a) \geq S(b+c) = 36bc(b^2+c^2) + 33(b^2-c^2)^2 - 5(b+c)^4 + 2bc(b+c)^2.$$

Everything now becomes clear. We have of course

$$\begin{aligned} S(b+c) &= 33(b^2-c^2)^2 - 5(b^4+c^4) + 30b^2c^2 + 16bc(b^2+c^2) + 2bc(b+c)^2 \\ &= 28(b^2-c^2)^2 + 16bc(b-c)^2 + 2bc(b-c)^2 \geq 0. \end{aligned}$$

Therefore $S \geq 0$ in all cases and by **SOS** method, we get the desired result. The equality holds for $a = b = c$ and $a = 2b = 2c$ up to permutation.

∇

Example 1.5.4. *Prove that if a, b, c are the side lengths of a triangle then*

$$9F(a,b,c) \geq 2F(a-b, b-c, c-a),$$

and if a, b, c are the side lengths of an acute triangle then

$$3F(a,b,c) \geq F(a-b, b-c, c-a).$$

(Pham Kim Hung)

SOLUTION. We will only prove the second part of this problem because the first part can be deduced similarly but simpler. Now suppose that a, b, c are side lengths of an acute triangle. Clearly, if $x+y+z=0$ then

$$x^3 + y^3 + z^3 + 3xyz - xy(y+x) - yz(y+z) - zx(z+x) = 9xyz.$$

Then, the inequality is equivalent to

$$\sum_{cyc}(a+b-c)(a-b)^2 \geq 3(a-b)(b-c)(c-a).$$

It's possible to assume that $a \geq c \geq b$. By the mixing all variables method, we conclude that it's sufficient to prove the inequality in case a, b, c are the side lengths of a right triangle (that means $a^2 = b^2 + c^2$). Because of the homogeneity, we can assume that $a = 1$ and $b^2 + c^2 = 1$. The inequality is reduced to

$$bc(3 - 2b - 2c) \geq 3(1 - b - c + bc)(c - b)$$

$$\Leftrightarrow\ 3(b + c - 1)(c - b) \geq bc(5c - b - 3).$$

Because $b + c \leq \sqrt{2(b^2 + c^2)} = \sqrt{2}$, we deduce that

$$\frac{b + c - 1}{bc} = \frac{b + c - \sqrt{b^2 + c^2}}{bc} = \frac{2}{b + c + \sqrt{b^2 + c^2}} \geq \frac{2}{1 + \sqrt{2}} = 2\left(\sqrt{2} - 1\right).$$

and it remains to prove that

$$6\left(\sqrt{2} - 1\right)(c - b) \geq 5c - b - 3$$

$$\Leftrightarrow\ \left(11 - 6\sqrt{2}\right)c + \left(6\sqrt{2} - 7\right)b \leq 3.$$

This last inequality is an obvious application of **Cauchy-Schwarz** inequality

$$\left(11 - 6\sqrt{2}\right)c + \left(6\sqrt{2} - 7\right)b \leq \sqrt{\left(11 - 6\sqrt{2}\right)^2 + \left(6\sqrt{2} - 7\right)^2} \approx 2.9 < 3.$$

This ends the proof. The equality holds for the equilateral triangle, $a = b = c$.

∇

Example 1.5.5. *Let a, b, c be non-negative real numbers. Prove that*

$$9F(a^2, b^2, c^2) \geq 8F\left((a - b)^2, (b - c)^2, (c - a)^2\right).$$

(Pham Kim Hung)

SOLUTION. We use the mixing all variables method, similarly as in the preceding problem. We can assume that $a \geq b \geq c = 0$. In this case, we obtain

$$\begin{aligned}
F\left((a - b)^2, (b - c)^2, (c - a)^2\right) &= a^6 + b^6 + (a - b)^6 + 3a^2b^2(a - b)^2 - (a^2 + b^2)(a - b)^4 \\
&\quad - a^2b^2(a^2 + b^2) - (a - b)^2(a^4 + b^4) \\
&= (a - b)^2\left(4ab(a^2 + b^2) - (a^2 - b^2)^2 + (a - b)^4 + a^2b^2\right) \\
&= 8a^2b^2(a - b)^2.
\end{aligned}$$

Moreover, because $F(a^2, b^2, c^2) = 9(a - b)^2(a^2 + b^2)(a + b)^2$, it remains to prove that

$$9(a^2 + b^2)(a + b)^2 \geq 72a^2b^2,$$

which is obvious because $a^2 + b^2 \geq 2ab$ and $(a + b)^2 \geq 4ab$. The proof is finished and the equality holds for $a = b = c$ and $a = b, c = 0$ up to permutation.

∇

Article 3

Thought Brings Knowledge - Varied Ideas

1.6 Exponent Smash

Is there anything to say about the simplest inequalities such as $a^2 + b^2 + c^2 \geq ab + bc + ca$ or $3(a^2 + b^2 + c^2) \geq (a + b + c)^2 \geq 3(ab + bc + ca)$? In a particular situation, in an unusual situation, they become extremely complex, hard but interesting and wonderful as well. That's why I think that this kind of inequalities is very strange and exceptional.

The unusual situation we have already mentioned is when each variable a, b, c stands as the exponent of another number. Putting them in places of exponents, must have broken up the simple inequalities between variables mentioned above. Let's see some problems.

Example 1.6.1. *Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Find the maximum of the following expressions*

$$(a) \quad S_2 = 2^{ab} + 2^{bc} + 2^{ca}.$$
$$(b) \quad S_4 = 4^{ab} + 4^{bc} + 4^{ca}.$$

(Pham Kim Hung)

SOLUTION. Don't hurry to conclude that $\max S_2 = 6$ and $\max S_4 = 12$ because the reality is different. We figure out a solution by the mixing variable method and solve a general problem that involves both (a) and (b). WLOG, assume that $a \geq b \geq c$ and $k \geq 1$ is a positive real constant. Consider the following expression

$$f(a, b, c) = k^{ab} + k^{bc} + k^{ca}.$$

Let $t = \dfrac{a+b}{2}, u = \dfrac{a-b}{2}$, then $t \geq 1, a = t + u, b = t - u$ and

$$f(a, b, c) = k^{t^2 - u^2} + k^{c(t-u)} + k^{c(t+u)} = g(u).$$

Its derivative is

$$g'(u) = 2u \ln k \cdot k^{t^2-u^2} + \ln k \cdot ck^{ct}(k^{cu} - k^{-cu}).$$

By **Lagrange** theorem, there exists a real number $r \in [-u, u]$ such that

$$\frac{k^{cu} - k^{-cu}}{2u} = ck^{cr},$$

and therefore $k^{cu} - k^{-cu} \le 2uck^{cu}$ (because $r \le u$). Moreover, $c \le 1$ and $c(t+u) \le (t-u)(t+u) = t^2 - u^2$ so we obtain $g'(u) \le 0$. Thus

$$g(u) \le g(0) = k^{t^2} + 2k^{ct} = k^{t^2} + 2k^{t(3-2t)} = h(t).$$

We will prove that $h(t) \le \max\left(h\left(\frac{3}{2}\right), h(1)\right)$ $\forall k \ge 1$. Since

$$h'(t) = 2t \ln k \cdot k^{t^2} + 2(3-4t) \ln k \cdot k^{t(3-2t)}$$

we infer $h'(t) = 0 \Leftrightarrow 4t - 3 = t \cdot k^{3t(t-1)}$. Consider the following function

$$q(t) = 3t(t-1) \ln k - \ln(4t-3) + \ln t.$$

Because $q'(t) = (6t-3) \ln k - \dfrac{3}{t(4t-3)}$ is a decreasing function of t (when $t \ge 1$), we deduce that the equation $q'(t) = 0$ has no more than one root $t \ge 1$. According to **Rolle's** theorem, the equation $h'(t) = 0$ has no more than two roots $t \in \left[1, \frac{3}{2}\right]$. Moreover, because $h'(1) = 0$, we conclude that

$$h(t) \le \max\left(h(1), h\left(\frac{3}{2}\right)\right).$$

According to this proof, we can synthesize a general result as follows

★ *Let a, b, c be non-negative real numbers with sum 3. For all $k \ge 1$, we have*

$$k^{ab} + k^{bc} + k^{ca} \le \max\left(3k, k^{9/4} + 2\right).$$

∇

Example 1.6.2. *Let a, b, c be non-negative real numbers such that $a + b + c = 3$. Find the minimum of the expression*

$$3^{-a^2} + 3^{-b^2} + 3^{-c^2}.$$

(Pham Kim Hung)

SOLUTION. We will again propose and solve the general problem: for each real number $k > 0$, find the minimum of the following expression

$$P = k^{a^2} + k^{b^2} + k^{c^2}.$$

Certainly, if $k \geq 1$ then $P \geq 3k$ by **AM-GM** inequality. Therefore we only need to consider the remaining case $k \leq 1$. WLOG, assume that $a \geq b \geq c$. Let $t = \dfrac{a+b}{2}, u = \dfrac{a-b}{2}$, then $t \geq 1$ and $a = t+u, b = t-u$. Let $k' = \dfrac{1}{k} \geq 1$ and consider the following function

$$g(u) = k^{(t-u)^2} + k^{(t+u)^2} + k^{c^2}.$$

Since

$$g'(u) = 2\ln k \cdot (t+u)k^{(t+u)^2} - 2\ln k \cdot (t-u)k^{(t-u)^2},$$

we deduce that $g'(u) = 0 \Leftrightarrow \ln(t+u) - \ln(t-u) = -4tu \ln k$. Letting now

$$h(u) = \ln(t+u) - \ln(t-u) + 4tu \ln k,$$

we infer that

$$h'(u) = \frac{1}{t+u} + \frac{1}{t-u} + 4\ln k \cdot t = \frac{2t}{t^2-u^2} + 4t\ln k.$$

Therefore $h'(u) = 0 \Leftrightarrow 2(t^2-u^2)\ln k = -1 \Leftrightarrow 2ab\ln k' = 1$. Now we divide the problem into two smaller cases

(i) The first case. If $ab, bc, ca \leq \dfrac{1}{2\ln k'}$, then

$$k^{a^2} + k^{b^2} + k^{c^2} \geq k^{9/4} + 2k^{\frac{1}{2\ln k'}} = k^{9/4} + 2e^{-1/2}.$$

(ii) The second case. If $ab \geq \dfrac{1}{2\ln k'}$. From the previous result, we deduce that

$$h'(u) = 0 \Leftrightarrow u = 0 \Rightarrow g'(u) = 0 \Leftrightarrow u = 0,$$

therefore

$$g(u) \geq g(0) = 2k^{t^2} + k^{(3-2t)^2} = f(t).$$

Our remaining work is to find the minimum of $f(t)$ for $\dfrac{3}{2} \geq t \geq 1$. Since

$$f'(t) = 4\ln k\left(t.k^{t^2} - (3-2t).k^{(3-2t)^2}\right),$$

we refer that $f'(t) = 0 \Leftrightarrow (3-3t)(3-t)\ln k' = \ln(3-2t) - \ln t$. Denote

$$q(t) = (3-3t)(3-t)\ln k' - \ln(3-2t) - \ln t$$

then

$$q'(t) = (6t-12)\ln k' + \frac{2}{3-2t} + \frac{1}{t} = (6t-12)\ln k' + \frac{3}{t(3-2t)}.$$

In the range $\left[1, \frac{3}{2}\right]$, the function $t(3-2t)(2-t) = 2t^3-7t^2+6t$ is decreasing, hence the equation $q'(t) = 0$ has no more than one real root. By **Rolle's** theorem, the equation $f'(t)$ has no more than two roots in $\left[1, \frac{3}{2}\right]$. It's then easy to get that

$$f(t) \geq \min\left(f(1), f\left(\frac{3}{2}\right)\right) = \min\left(3k, 1+2k^{9/4}\right).$$

According to the previous solution, the following inequality holds

$$k^{a^2} + k^{b^2} + k^{c^2} \geq \min\left(3k, 1+2k^{9/4}, 2e^{-1/2} + k^{9/4}\right).$$

Notice that if $k \leq 1/3$ then

$$\min\left(3k, 1+2k^{9/4}, 2e^{-1/2} + k^{9/4}\right) = 3k.$$

Otherwise, if $k \geq 1/3$ then

$$1 + 2k^{9/4} \leq 2e^{-1/2} + k^{9/4}$$

$$\Rightarrow \min\left(3k, 1+2k^{9/4}, 2e^{-1/2} + k^{9/4}\right) = \min\left(3k, 1+2k^{9/4}\right).$$

Therefore, we can conclude that

$$k^{a^2} + k^{b^2} + k^{c^2} \geq \min\left(3k, 1+2k^{9/4}, 2e^{-1/2} + k^{9/4}\right) = \min\left(3k, 1+2k^{9/4}\right).$$

The initial problem is a special case for $k = \frac{1}{3}$. In this case, we have

$$3^{-a^2} + 3^{-b^2} + 3^{-c^2} \geq 1.$$

However, if $k = \frac{1}{2}$ then the following stranger inequality holds

$$2^{-a^2} + 2^{-b^2} + 2^{-c^2} \geq 1 + 2^{-5/4}.$$

∇

Example 1.6.3. *Let a, b, c be non-negative real numbers such that $a+b+c = 3$. Prove that*

$$2^{4ab} + 2^{4bc} + 2^{4ca} - 2^{3abc} \leq 513.$$

(Pham Kim Hung)

SOLUTION. In fact, this problem reminds us of **Schur** inequality only that we know have exponents (notice that if $a+b+c = 3$ then **Schur** inequality is equivalent to $4(ab+bc+ca) \leq 9+3abc$). According to the example 1.6.1, we deduce that

$$2^{4ab} + 2^{4bc} + 2^{4ca} = 16^{ab} + 16^{bc} + 16^{ca} \leq 16^{9/4} + 2 = 2^9 + 2,$$

moreover, we also have the obvious inequality $2^{3abc} \geq 1$, so

$$2^{4ab} + 2^{4bc} + 2^{4ca} - 2^{3abc} \leq 2^9 + 2 - 1 = 513.$$

$$\nabla$$

Transforming an usual inequality into one with exponents, you can obtain a new one. This simple idea leads to plenty inequalities, some nice, hard but also interesting. As a matter of fact, you will rarely encounter this kind of inequalities, however, I strongly believe that a lot of enjoyable, enigmatic matters acan be found here. So why don't you try it yourself?

1.7 Unexpected Equalities

Some people often make mistakes when they believe that all symmetric inequalities of three variables (in fraction forms) have their equality just in one of two standard cases: $a = b = c$ or $a = b, c = 0$ (and permutations of course). Sure almost all inequalities belong to this kind, but some are stranger. These inequalities, very few of them in comparison, make up a different and interesting area, where the usual **SOS** method is nearly impossible. Here are some examples.

Example 1.7.1. *Let x, y, z be non-negative real numbers. Prove that*

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{25(xy+yz+zx)}{(x+y+z)^2} \geq \ 8.$$

(Pham Kim Hung)

SOLUTION. WLOG, assume that $x \geq y \geq z$. Denote

$$f(x,y,z) = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{25(xy+yz+zx)}{(x+y+z)^2}.$$

We infer that

$$f(x,y,z) - f(x,y+z,0) = \frac{y}{z+x} + \frac{z}{x+y} - \frac{y+z}{x} - \frac{25yz}{(x+y+z)^2}$$

$$= yz\left(\frac{25}{(x+y+z)^2} - \frac{1}{x(x+y)} - \frac{1}{x(x+z)}\right).$$

Notice that $x \geq y \geq z$, so we have

$$\frac{z}{x+y} + \frac{y}{x+z} \leq \frac{z}{y+z} + \frac{y}{y+z} = 1$$

$$\Rightarrow \frac{1}{x+y} + \frac{1}{x+z} \leq \frac{3}{x+y+z}$$

$$\Rightarrow \frac{1}{x(x+y)} + \frac{1}{x(x+z)} \le \frac{3}{x(x+y+z)} \le \frac{25}{(x+y+z)^2}.$$

This shows that $f(x,y,z) \ge f(x,y+z,0)$. Denote $t = y+z$, then we have

$$f(x,y+z,0) = \frac{x}{t} + \frac{t}{x} + \frac{25xt}{(x+t)^2} = -2 + \frac{(x+t)^2}{xt} + \frac{25xt}{x+t)^2} \ge 8,$$

and the conclusion follows. The equality holds for $(x+t)^2 = 5xt$ or $\frac{x}{t} = \frac{-3 \pm \sqrt{5}}{2}$. In the initial inequality, the equality holds for $(x,y,z) \sim \left(\frac{-3 \pm \sqrt{5}}{2}, 1, 0\right)$.

Comment. In general, the following inequality can be proved by the same method

★ *Given non-negative real numbers x, y, z. For all $k \ge 16$, prove that*

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{k(xy+yz+zx)}{(x+y+z)^2} \ge\ 2\left(\sqrt{k} - 1\right).$$

$$\nabla$$

Example 1.7.2. *Let x, y, z be non-negative real numbers. Prove that*

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{16(xy+yz+zx)}{x^2+y^2+z^2} \ge\ 8.$$

(Tan Pham Van)

SOLUTION. We use mixing variables to solve this problem. Denote $x = \max\{x,y,z\}$ and

$$f(x,y,z) = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} + \frac{16(xy+yz+zx)}{x^2+y^2+z^2}.$$

It is easy to see that the statament $f(x,y,z) \ge\ f(x,y+z,0)$ is equivalent to

$$16(x+y+z)^2 x(x+z)(x+y) \ge\ (2x+y+z)\left(x^2+y^2+z^2\right)\left(x^2+(y+z)^2\right)$$

Notice that $x = \max\{x,y,z\}$, so

$$(x+y)(x+z) \ge\ \left(x^2+y^2+z^2\right)\ ;$$

$$16(x+y+z)^2 x \ge\ (2x+y+z)\left(x^2+(y+z)^2\right)\ ;$$

These two results show that $f(x,y,z) \ge\ f(x,y+z,0)$. Normalizing $x+y+z=1$, we get

$$f(x,y,z) \ge f(x,y+z,0) = f(x,1-x,0) = g(x).$$

We have

$$g'(x) = \frac{-(2x-1)(2x^2-2x-1)(6x^2-6x+1)}{x^2(x-1)^2(2x^2-2x+1)^2}$$

notice that $\frac{1}{3} \le x \le 1$, and it follows that the equation $g'(x) = 0$ has two roots

$$x \in \left\{ \frac{1}{2}\,;\ \frac{3+\sqrt{3}}{6} \right\}.$$

It's then easy to infer that

$$g(x) \ge g\left(\frac{3+\sqrt{3}}{6}\right) = 4,$$

as desired. The equality holds for $(x, y, z) \sim \left(3+\sqrt{3}\,;\ 3-\sqrt{3}\,;0\right)$.

∇

Example 1.7.3. *Let a, b, c be non-negative real numbers with sum 1. Prove that*

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} + \frac{27(a+b+c)^2}{a^2+b^2+c^2} \ge 52.$$

(Pham Kim Hung)

SOLUTION. WLOG, assume that $a \ge b \ge c$. Denote

$$f(a,b,c) = \frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} + \frac{27(a+b+c)^2}{a^2+b^2+c^2}.$$

We will prove that $f(a,b,c) \ge f(a, \sqrt{b^2+c^2}, 0)$. Indeed

$$f(a,b,c) - f\left(a, \sqrt{b^2+c^2}, 0\right)$$

$$= \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} - \frac{b^2+c^2}{a^2} + \frac{27(a+b+c)^2 - 27\left(a+\sqrt{b^2+c^2}\right)^2}{a^2+b^2+c^2}$$

$$\ge -b^2c^2\left(\frac{1}{a^2(a^2+b^2)} + \frac{1}{a^2(a^2+c^2)}\right) + \frac{54a\left(b+c-\sqrt{b^2+c^2}\right)}{a^2+b^2+c^2}$$

$$\ge -b^2c^2\left(\frac{1}{a^2(a^2+b^2)} + \frac{1}{a^2(a^2+c^2)}\right) + \frac{54b^2c^2}{4bc(a^2+b^2+c^2)}$$

Moreover, because

$$\frac{3}{a^2} \ge \frac{54}{4bc}\,;$$

$$\frac{1}{a^2(a^2+b^2)} + \frac{1}{a^2(a^2+c^2)} \le \frac{3}{a^2+b^2+c^2}\,;$$

We infer that $f(a,b,c) \ge f\left(a, \sqrt{b^2+c^2}, 0\right)$. Denote $t = \sqrt{b^2+c^2}$, then we have

$$f(a,t,0) = \frac{a^2}{t^2} + \frac{t^2}{a^2} + \frac{27(a+t)^2}{a^2+t^2} = -2 + \frac{(a^2+t^2)^2}{a^2t^2} + \frac{54at}{a^2+t^2} + 27$$

$$= \frac{(a^2+t^2)^2}{a^2t^2} + \frac{27at}{a^2+t^2} + \frac{27at}{a^2+t^2} + 25 \geq 3\sqrt[3]{27\cdot 27} + 25 = 52.$$

This ends the proof. The equality holds for $(a,b,c) \sim \left(\frac{-3\pm\sqrt{5}}{2}, 1, 0\right)$.

Comment. In a similar way, we can prove the following general inequality

★ *Given non-negative real numbers x, y, z. For all $k \geq 8$, prove that*

$$\frac{a^2}{b^2+c^2} + \frac{b^2}{c^2+a^2} + \frac{c^2}{a^2+b^2} + \frac{k(a+b+c)^2}{a^2+b^2+c^2} \geq k + \sqrt[3]{k} - 2.$$

∇

Example 1.7.4. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{24}{(a+b+c)^2} \geq \frac{8}{ab+bc+ca}.$$

(Pham Kim Hung)

SOLUTION. WLOG, assume that $a \geq b \geq c$. Denote

$$f(a,b,c) = \frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} + \frac{24}{(a+b+c)^2} - \frac{8}{ab+bc+ca}.$$

We get that

$$\begin{aligned}
&f(a,b,c) - f(a,b+c,0) \\
&= \frac{1}{(a+b)^2} - \frac{1}{(a+b+c)^2} + \frac{1}{(c+a)^2} - \frac{1}{a^2} - \frac{8}{ab+bc+ca} + \frac{8}{a(b+c)} \\
&\geq \frac{8bc}{a(b+c)(ab+bc+ca)} - \frac{c(2a+c)}{a^2(a+c)^2}.
\end{aligned}$$

Because $a \geq b \geq c$, we get that

$$8ab \geq (2a+c)(b+c)\ ;$$

$$(a+c)^2 \geq (ab+bc+ca)\ ;$$

These two results show that $f(a,b,c) - f(a,b+c,0) \geq 0$. Denote $t = b+c$, then we get

$$f(a,b+c,0) = f(a,t,0) = \frac{1}{a^2} + \frac{1}{t^2} + \frac{25}{(a+t)^2} - \frac{8}{at}.$$

By **AM-GM** inequality, we have

$$at\left(\frac{1}{a^2} + \frac{1}{t^2} + \frac{25}{(a+t)^2}\right) = -2 + \frac{(a+t)^2}{at} + \frac{25}{(a+t)^2} \geq -2 + 2\sqrt{25} = 8.$$

This ends the proof. The equality holds for $(a,b,c) \sim \left(\frac{-3\pm\sqrt{5}}{2}, 1, 0\right)$.

Comment. By a similar method, we can prove the following generalization

★ *Given non-negative real numbers a,b,c. For all $k \geq 15$, prove that*

$$\frac{1}{(a+b)^2}+\frac{1}{(b+c)^2}+\frac{1}{(c+a)^2}+\frac{k}{(a+b+c)^2} \geq \frac{2\sqrt{k+1}-2}{ab+bc+ca}.$$

Moreover, the following result can also be proved similarly.

★ *Given non-negative real numbers a,b,c. Prove that*

$$\frac{1}{a^2+b^2}+\frac{1}{b^2+c^2}+\frac{1}{c^2+a^2}+\frac{8}{a^2+b^2+c^2} \geq \frac{6}{ab+bc+ca}.$$

∇

Example 1.7.5. *Let a,b,c be non-negative real numbers. Prove that*

$$\frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4(a+b+c)(ab+bc+ca)}{a^3+b^3+c^3} \geq 5.$$

(Tan Pham Van)

SOLUTION. WLOG, assume that $a \geq b \geq c$. Denote $t = b+c$ and

$$f(a,b,c) = \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}+\frac{4(a+b+c)(ab+bc+ca)}{a^3+b^3+c^3}.$$

We have

$$\begin{aligned}
&f(a,b,c)-f(a,b+c,0)\\
&=\frac{b}{c+a}+\frac{c}{a+b}-\frac{b+c}{a}+\frac{4(a+b+c)(ab+bc+ca)}{a^3+b^3+c^3}-\frac{4a(b+c)}{a^2-a(b+c)+(b+c)^2}\\
&=\frac{4(a^3+3a(b+c)^2+(b+c)^3)bc}{(a^2-a(b+c)+(b+c)^2)(a^3+b^3+c^3)}-\frac{(2a+b+c)bc}{a(a+b)(a+c)} \geq 0
\end{aligned}$$

because

$$(a+b)(a+c) \geq \ a^2-a(b+c)+(b+c)^2\ ;$$
$$4a(a^3+3a(b+c)^2+(b+c)^3) \geq \ (2a+b+c)(a^3+b^3+c^3)\ .$$

Moreover, by **AM-GM** inequality, we have

$$\begin{aligned}
f(a,b+c,0) = f(a,t,0) &= \frac{a}{t}+\frac{t}{a}+\frac{4at}{a^2-at+t^2}\\
&= \frac{a^2-at+t^2}{at}+\frac{4at}{a^2-at+t^2}+1 \geq 5.
\end{aligned}$$

This is the end of the proof. The equality holds for $(a,b,c) \sim \left(\frac{3\pm\sqrt{5}}{2}, 1, 0\right)$.

∇

Example 1.7.6. *Let a, b, c be non-negative real numbers. Prove that*

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}+\frac{9\sqrt{ab+bc+ca}}{a+b+c}\geq 6.$$

(Pham Kim Hung)

SOLUTION. WLOG, assume that $a\geq b\geq c$. We have

$$\sqrt{\frac{ab}{a+c}}+\sqrt{\frac{ac}{a+b}}\geq\sqrt{\frac{b\cdot b}{b+c}}+\sqrt{\frac{c\cdot c}{c+b}}=\sqrt{b+c}$$

$$\Rightarrow\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}\geq\sqrt{\frac{b+c}{a}}.$$

Let now $t=b+c$, then we get that

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}+\frac{9\sqrt{ab+bc+ca}}{a+b+c}$$

$$\geq\sqrt{\frac{a}{t}}+\sqrt{\frac{t}{a}}+\frac{9\sqrt{at}}{a+t}=\frac{a+t}{\sqrt{at}}+\frac{9\sqrt{at}}{a+t}\geq 6.$$

The equality holds for $a+t=3\sqrt{at}$ or $(a,b,c)\sim\left(\dfrac{7\pm\sqrt{45}}{2},1,0\right)$.

Comment. In a similar way, we can prove the following general result

★ *Given non-negative real numbers a, b, c prove for all $k\geq 4$* that

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}+\frac{k\sqrt{ab+bc+ca}}{a+b+c}\geq 2\sqrt{k}.$$

The following result is also true and can be proved by a similar method

★ *Given non-negative real numbers a, b, c, prove for all $k\geq\dfrac{3\sqrt{3}}{2}$ that*

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}+\frac{k(a+b+c)}{\sqrt{a^2+b^2+c^2}}\geq\sqrt{1+\sqrt[3]{4k^2}}+k\sqrt{\frac{1+\sqrt[3]{4k^2}}{-1+\sqrt[3]{4k^2}}}.$$

Letting $k=4$, we get the following result

★ *Given non-negative real numbers a, b, c prove that*

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}+\frac{4(a+b+c)}{\sqrt{a^2+b^2+c^2}}\geq\frac{37}{5}.$$

∇

Example 1.7.7. *Let a, b, c be non-negative real numbers. Prove that*

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}+\sqrt{\frac{27(ab+bc+ca)}{a^2+b^2+c^2}} \ge \frac{7\sqrt{2}}{2}.$$

(Vo Quoc Ba Can)

SOLUTION. Similarly as in the previous problem, we get that if $a \ge b \ge c$ and $t = b+c$ then

$$\sqrt{\frac{a}{b+c}}+\sqrt{\frac{b}{c+a}}+\sqrt{\frac{c}{a+b}}+\sqrt{\frac{27(ab+bc+ca)}{a^2+b^2+c^2}} \ge \frac{a+t}{\sqrt{at}}+3\sqrt{3}\cdot\sqrt{\frac{at}{a^2+t^2}}.$$

Denote $x = \dfrac{a+t}{\sqrt{at}} \ge 2$. It remains to prove that

$$f(x) = x + \frac{3\sqrt{3}}{\sqrt{x^2-2}} \ge \frac{7\sqrt{2}}{2}.$$

Checking the derivative $f'(x)$, it is easy to see that the equation $f'(x) = 0$ has exactly one root $x = 2\sqrt{2}$, therefore

$$f(x) \ge f\left(2\sqrt{2}\right) = 2\sqrt{2} + \frac{3\sqrt{3}}{\sqrt{6}} = \frac{7\sqrt{2}}{2}.$$

The equality holds for $a+t = 2\sqrt{2at}$ or $(a,b,c) \sim \left(3 \pm \sqrt{8}, 1, 0\right)$.

∇

Example 1.7.8. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{8}{a+b+c} \ge \frac{6}{\sqrt{ab+bc+ca}}.$$

(Pham Kim Hung)

SOLUTION. Similarly to the previous proofs, we assume first that $a \ge b \ge c$ and denote

$$f(a,b,c) = \frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}+\frac{8}{a+b+c}-\frac{6}{\sqrt{ab+bc+ca}}.$$

Let $t = b+c$, then we have

$$f(a,b,c) - f(a,t,0) = \frac{1}{a+b}+\frac{1}{a+c}-\frac{1}{a+b+c}-\frac{1}{a}-\frac{6}{\sqrt{ab+bc+ca}}+\frac{6}{\sqrt{a(b+c)}}$$

$$\ge -\frac{c}{a(a+c)}+\frac{6\left(\sqrt{ab+bc+ca}-\sqrt{ab+ac}\right)}{\sqrt{a(b+c)(ab+bc+ca)}}.$$

Using $ab+bc+ca \le (a+c)^2$ and $a(b+c) \le 2ab$, we get

$$\frac{\sqrt{ab+bc+ca}-\sqrt{ab+ac}}{\sqrt{a(b+c)(ab+bc+ca)}} = \frac{bc}{\sqrt{a(b+c)}(ab+bc+ca)+a(b+c)\sqrt{ab+bc+ca}}$$

$$\ge \frac{bc}{3ab\sqrt{a(b+c)}+2ab(a+c)} = \frac{c}{a\left(3\sqrt{a(b+c)}+2(a+c)\right)}.$$

Therefore

$$f(a,b,c)-f(a,t,0) \ge \frac{6c}{a\left(3\sqrt{a(b+c)}+2(a+c)\right)} - \frac{c}{a(a+c)}.$$

Moreover, since $a \ge b \ge c$, we infer that

$$6(a+c) = 4(a+c)+2(a+c) \ge \frac{3}{2}(a+b+c)+2(a+c) \ge 3\sqrt{a(b+c)}+2(a+c),$$

which means that $f(a,b,c) \ge f(a,t,0)$. Furthermore, by **AM-GM** inequality, we conclude

$$f(a,t,0) = \frac{9}{a+t}+\frac{1}{a}+\frac{1}{t}-\frac{6}{\sqrt{at}}$$

$$= \frac{1}{\sqrt{at}}\left(\frac{9\sqrt{at}}{a+t}+\frac{a+t}{\sqrt{at}}-6\right) \ge 0.$$

This ends the proof. The equality holds for $a+t = 3\sqrt{at}$ or $(a,b,c) \sim \left(\frac{7\pm\sqrt{45}}{2},1,0\right)$.

∇

1.8 Undesirable Conditions

We will consider now some symmetric inequalities different from every other inequalities we used to solve before. In these problems, variables are restricted by particular conditions which can't have the solution $a = b = c$ (so $a = b = c$ can not make up any case of equality). The common technique is to solve them using special expressions and setting up equations involving them.

Example 1.8.1. *Suppose that a, b, c are three positive real numbers satisfying*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 13.$$

Find the minimum value of

$$P = (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(Vasile Cirtoaje)

SOLUTION. Let now

$$x = \frac{a}{b}+\frac{b}{c}+\frac{c}{a},\ y = \frac{b}{a}+\frac{c}{b}+\frac{a}{c},$$

then we obtain that

$$x^2 = 2y + \sum_{cyc}\frac{a^2}{b^2}\ ;\ y^2 = 2x + \sum_{cyc}\frac{b^2}{a^2}\ ;$$

Because $x + y = 10$, **AM-GM** inequality yields that

$$P - 3 = x^2 + y^2 - 2(x+y) \geq \frac{1}{2}(x+y)^2 - 2(x+y) = 50 - 20 = 30$$

therefore $P \geq 33$ and $\min P = 33$ with equality for

$$x = y \ \Leftrightarrow\ \sum_{cyc}\frac{a}{b} = \sum_{cyc}\frac{b}{a}\ \Leftrightarrow\ (a-b)(b-c)(c-a) = 0,$$

combined with the hypothesis $(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 10$, we conclude that $P = 33$ if and only if $(a, b, c) \sim \left(2 \pm \sqrt{3}, 1, 1\right)$ and permutations.

Comment. This approach is still effective for the general problem where 13 is replaced by an arbitrary real number $k \geq 9$.

∇

Example 1.8.2. *Let a, b, c be three positive real numbers such that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 16.$$

Find the maximum value of

$$P = (a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right).$$

(Vasile Cirtoaje and Pham Kim Hung)

SOLUTION. **First Solution.** Taking into account an example in Volume I, we deduce that

$$\frac{13-\sqrt{5}}{2} \le \sum_{cyc}\frac{a}{b} \le \frac{13+\sqrt{5}}{2}\ ;$$

$$\frac{13-\sqrt{5}}{2} \le \sum_{cyc}\frac{b}{a} \le \frac{13+\sqrt{5}}{2}\ .$$

therefore

$$\left(\sum_{cyc}\frac{a}{b}-\sum_{cyc}\frac{b}{a}\right)^2+\left(\sum_{cyc}\frac{a}{b}+\sum_{cyc}\frac{b}{a}\right)^2 \le 5+13^2=174$$

$$\Rightarrow \left(\sum_{cyc}\frac{a}{b}\right)^2+\left(\sum_{cyc}\frac{b}{a}\right)^2 \le 87\ \Rightarrow \sum_{cyc}\frac{a^2}{b^2}+\sum_{cyc}\frac{b^2}{a^2} \le 61$$

$$\Rightarrow \left(\sum_{cyc}a^2\right)\left(\sum_{cyc}\frac{1}{a^2}\right) \le 64.$$

The maximum of P is 64, attained for $\dfrac{a}{b}=\dfrac{b}{c}=\dfrac{3\pm\sqrt{5}}{2}$ or any permutation.

Second Solution. We denote $x=\sum\limits_{cyc}\dfrac{a}{b},\ y=\sum\limits_{cyc}\dfrac{b}{a},\ m=\sum\limits_{cyc}\dfrac{a^2}{bc}$ and $n=\sum\limits_{cyc}\dfrac{bc}{a^2}$. We certainly have $x+y=13$ and $xy=3+m+n$. Moreover

$$x^3+y^3=\sum_{sym}\frac{a^3}{b^3}+6(m+n)+12\ (\star)$$

By **AM-GM** inequality, we have

$$(m+n)^2 \ge 4mn = 4\left(3+\sum_{sym}\frac{a^3}{b^3}\right).$$

Combining this result with $(\star)$, we deduce that

$$(m+n)^2 \ge 12+4\left(x^3+y^3-6(m+n)-12\right)$$

Because $x+y=13$, we obtain

$$x^3+y^3=(x+y)^3-3xy(x+y)=13^3-39(3+m+n)=2080-39(m+n)$$

This yields that (with $t = m + n$)

$$t^2 \geq -36 + 4(2080 - 39t) - 24t \Rightarrow t^2 + 180t - 8284 \geq 0 \Rightarrow t \geq 38.$$

Therefore $m + n \geq 38$ or $xy \geq 41$. This result implies

$$\begin{aligned}(a^2+b^2+c^2)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right) &= 3 + x^2 + y^2 - 2(x+y) \\ &= (x+y)^2 - 2(x+y) + 3 - 2xy \\ &\leq 13^2 - 2\cdot 13 + 3 - 2\cdot 41 = 64.\end{aligned}$$

The equality holds if and only if $m = n$, or

$$\sum_{cyc} \frac{a^2}{bc} = \sum_{cyc} \frac{bc}{a^2} \Leftrightarrow (a^2 - bc)(b^2 - ca)(c^2 - ab) = 0,$$

and in this case, it's easy to conclude that the maximum of P is 64. The equality holds if and only if $\frac{a}{b} = \frac{b}{c} = \frac{3 \pm \sqrt{5}}{2}$ up to permutation.

Comment. Both solutions above can still be used to solve the general problem in which 16 is replaced by an arbitrary real number $k \geq 9$.

∇

Example 1.8.3. *Suppose that a, b, c are three positive real numbers satisfying that*

$$(a+b+c)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right) = 16.$$

Find the minimum and maximum value of

$$P = (a^4+b^4+c^4)\left(\frac{1}{a^4}+\frac{1}{b^4}+\frac{1}{c^4}\right).$$

SOLUTION. We denote x, y, m, n as in the previous problem and also denote

$$p = \sum_{cyc} \frac{a^2}{b^2}\ ;\ q = \sum_{cyc} \frac{b^2}{a^2}\ ;$$

The expression P can be rewritten as

$$P = 3 + \sum_{cyc} \frac{a^4}{b^4} = 3 + p^2 + q^2 - 2(p+q).$$

The hypothesis yields that $x + y = 13$. Moreover, we have

$$xy = 3 + m + n\ (1)\ ;\ \ x^2 + y^2 = p + q + 2(x+y)\ (2)\ ;$$

$$p^2+q^2=\sum_{cyc}\frac{a^4}{b^4}+2(p+q)\ (3)\ ;\ \ pq=3+\sum_{cyc}\frac{a^4}{b^2c^2}+\sum_{cyc}\frac{b^2c^2}{a^4}\ (4)\ ;$$

$$m^2+n^2=\sum_{cyc}\frac{a^4}{b^2c^2}+\sum_{cyc}\frac{b^2c^2}{a^4}+2(m+n)\ (5)\ ;\ \ x^3+y^3=mn+6(m+n)+9\ (6)\ .$$

The two results (4) and (5) combined show that

$$pq=3+m^2+n^2-2(m+n)\ (7)\ ;$$

According to (1) and (2) and noticing that $x+y=13$, we have

$$p+q=13^2-2.13-2(3+m+n)=137-2(m+n)$$

The equation (6) shows that

$$13^3-39(3+m+n)=mn+6(m+n)+9\ \Rightarrow mn=2071-45(m+n)$$

$$\Rightarrow pq=3+(m+n)^2-2mn-2(m+n)=t^2+88t-4139$$

where $t=m+n$. Therefore P can be expressed as a function of t as

$$\begin{aligned}P=2+(p+q-1)^2-2pq&=2+4(68-t)^2-2(t^2+88t-4139)\\&=2t^2-720t+26776=f(t)\end{aligned}$$

Taking into account the preceding problem, we obtain $39.25\geq t=m+n\geq 38$ and the conclusion follows: the minimum of P is $f(39.25)=\frac{12777}{8}$, with equality for $a=b=\frac{(11\pm\sqrt{105})c}{4}$ or permutations; the maximum of P is $f(38)=2304$, with equality for $\frac{a}{b}=\frac{b}{c}=\frac{3\pm\sqrt{5}}{2}$ or permutations.

∇

Example 1.8.4. *Let a,b,c,d be positive real numbers such that*

$$(a+b+c+d)\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)=20.$$

Prove that

$$\left(a^2+b^2+c^2+d^2\right)\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}+\frac{1}{d^2}\right)\geq 36.$$

(Vasile Cirtoaje, Pham Kim Hung, Phan Thanh Nam, **VMEO 2006**)

SOLUTION. For each triple of positive real numbers (x,y,z), we denote

$$F(x,y,z)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x};$$

Clearly

$$F(x,y,z)^2 = F(x^2,y^2,z^2) + 2F(z,y,x)\ ;\ F(z,y,x)^2 = F(z^2,y^2,x^2) + 2F(x,y,z)\ .$$

The condition of the problem can be rewritten as

$$\sum_{cyc} (F(a,b,c) + F(c,b,a)) = 32.$$

Our inequality is equivalent to

$$\sum_{cyc} \left(F(a^2,b^2,c^2) + F(c^2,b^2,a^2)\right) \geq 64$$

$$\Leftrightarrow \sum_{cyc} \left(F(a,b,c)^2 - 2F(c,b,a) + F(c,b,a) - 2F(a,b,c)\right) \geq 64$$

$$\Leftrightarrow \sum_{cyc} \left(F^2(a,b,c) + F^2(c,b,a)\right) \geq 128.$$

Applying **Cauchy-Schwarz** inequality, we conclude

$$\sum_{cyc} \left(F^2(a,b,c) + F^2(c,b,a)\right) \geq \frac{1}{8}\left(\sum_{cyc} (F(a,b,c) + F(c,b,a))\right)^2 = \frac{32^2}{8} = 128,$$

and the conclusion follows. The equality holds for

$$F(a,b,c) = F(c,b,a) = F(a,c,d) = F(d,c,a) = F(b,c,d) = F(d,c,b) = 4,$$

or equivalently $(a,b,c,d) \sim \left(\frac{3 \pm \sqrt{5}}{2}\ ;\ \frac{3 \pm \sqrt{5}}{2}\ ;\ 1\ ;1\right)$ and every permutation.

∇

Example 1.8.5. *Suppose that a, b, c are three positive real numbers satisfying*

$$\left(a^2 + b^2 + c^2\right)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 = 36.$$

Prove the following inequality

$$(a+b+c)^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) > 34.$$

(Pham Kim Hung)

SOLUTION. As in the preceding problems, we denote

$$x = \sum_{cyc} \frac{a}{b}\ ;\ y = \sum_{cyc} \frac{b}{a}\ ;\ m = \sum_{cyc} \frac{a^2}{bc}\ ;\ n = \sum_{cyc} \frac{bc}{a^2}\ ;$$

The hypothesis shows that

$$\sum_{cyc} \frac{a^2}{b^2} + \sum_{cyc} \frac{b^2}{a^2} + 2(x+y) + 2m = 33 \ \Rightarrow x^2 + y^2 + 2m = 33.$$

We also have two relations (similar to the relations in the previous problem)

$$xy = 3 + m + n,\ x^3 + y^3 = mn + 6(m+n) + 12$$

and therefore

$$x + y = \sqrt{x^2 + y^2 + 2xy} = \sqrt{39 + 2n}$$

$$\Rightarrow mn + 6(m+n) + 12 = \sqrt{39+2n}(30 - n - 3m).$$

Denoting $r = \sqrt{39+2n}$, it follows that $2n = r^2 - 39$ and

$$m(r^2 - 39) + 12m + 6(r^2 - 39) + 24 = r(99 - r^2 - 6m)$$

$$\Rightarrow m = \frac{210 + 99r - 6r^2 - r^3}{r^2 + 6r - 27}$$

$$\Rightarrow m - n = \frac{210 + 99r - 6r^2 - r^3}{r^2 + 6r - 27} - \frac{r^2 - 39}{2} = f(r).$$

By **Schur** inequality, we have $n + 3 \ge x + y$. Therefore

$$n + 3 \ge \sqrt{39 + 2n} \ \Rightarrow n^2 + 4n - 30 \ge 0 \ \Rightarrow n \ge -2 + \sqrt{34} \ \Rightarrow r \ge \sqrt{35 + 2\sqrt{34}}.$$

On the other hand, because $m - n = f(r)$ is a decreasing function of r, we conclude

$$m - n \le f\left(\sqrt{35 + 2\sqrt{34}}\right) < 1$$

and

$$\left(a^2 + b^2 + c^2\right)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2 - (a+b+c)^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) = 2(m-n) \le 2$$

$$\Rightarrow (a+b+c)^2\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) > 34.$$

∇

Article 4

Cyclic Inequalities of Degree 3

1.9 Getting Started

Hoo Joo Lee proposed the following result a long time ago (known as "Symmetric inequality of Degree 3" theorem, or **SD3** theorem)

Theorem 2 (SD3). *Let P(a,b,c) be a symmetric polynomial of degree* 3. *The following conditions are equivalent to each other*

- $P(1,1,1), P(1,1,0), P(1,0,0) \geq 0.$
- $P(a,b,c) \geq 0 \; \forall a,b,c \geq 0.$

Although this theorem is quite strong, it is restricted to the field of symmetric inequalities. With the help of our global derivative and the mixing all variables method, we will figure out a more general formula to check cyclic inequalities which have degree 3. It will be called "Cyclic inequality of Degree 3" theorem, or **CD3** theorem,

Theorem 3 (CD3). *Let $P(a,b,c)$ be a cyclic homogeneous polynomial of degree* 3. *The inequality $P \geq 0$ holds for all non-negative variables a,b,c if and only if*

$$P(1,1,1) \geq 0 \; ; \; P(a,b,0) \geq 0 \; \forall a,b \geq 0;$$

PROOF. The necessary condition is obvious. We only need to consider the sufficient condition. Assume that

$$P(1,1,1) \geq 0 \; ; \; P(a,b,0) \geq 0 \; \forall a,b \geq 0;$$

We will prove that for all $a,b,c \geq 0$ we have

$$P(a,b,c) = m\sum_{cyc} a^3 + n\sum_{cyc} a^2b + p\sum_{cyc} ab^2 + qabc \geq 0.$$

The condition $P(1,1,1) \geq 0$ yields that

$$m \geq 0 \; ; \; 3m + 3n + 3p + q \geq 0 \; ;$$

The condition $P(a,b,0) \geq 0\ \forall a,b \geq 0$ yields that (by choosing $a = b = 1$ and $a = 1, b = 0$)

$$P(1,0,0) = m \geq 0\ ;\ P(1,1,0) = 2m + n + p \geq 0\ .$$

Consider the inequality

$$m\sum_{cyc} a^3 + n\sum_{cyc} a^2b + p\sum_{cyc} ab^2 + qabc \geq 0.$$

Taking the global derivative, we get

$$3m\left(\sum_{cyc} a^2\right) + n\left(\sum_{cyc} a^2 + 2\sum_{cyc} ab\right) + p\left(\sum_{cyc} b^2 + 2\sum_{cyc} ab\right) + \left(q\sum_{cyc} ab\right) \geq 0$$

or

$$(3m + n + p)\sum_{cyc} a^2 + (2n + 2p + q)\sum_{cyc} ab \geq 0.$$

Notice that $3m + n + p = m + (2m + n + p) \geq 0$ and $(3m + n + p) + (2n + 2p + q) = 3m + 3n + 3p + q \geq 0$, so we deduce that

$$(3m + n + p)\sum_{cyc} a^2 + (2n + 2p + q)\sum_{cyc} ab \geq (3m + n + p)\left(\sum_{cyc} a^2 - \sum_{cyc} ab\right) \geq 0.$$

According to the principle of the global derivative, the inequality $P(a,b,c) \geq 0$ holds if and only if it holds when $\min\{a,b,c\} = 0$. Because the inequality is cyclic, we may assume that $c = 0$. The conclusion follows immediately since $P(a,b,0) \geq 0\ \forall a,b \geq 0$.

Comment. 1. Hoo Joo Lee's theorem can be regarded as a direct corollary of this theorem for the symmetric case. Indeed, if $n = p$, the inequality

$$P(a,b,0) = m(a^3 + b^3) + n(a^2b + b^2a) \geq 0$$

holds for all $a, b \geq 0$ if and only if $m + n \geq 0$ (this property is simple). Notice that $m + n = P(1,1,0)$, so we get Ho Joo Lee's inequality.

2. According to this theorem, we can conclude that, in order to check an arbitrary cyclic inequality which has degree 3, it suffices to check it in two cases, when all variables are equal and when one variable is 0. For the inequality $F(a,b,0) \geq 0$, we can let $b = 1$ and change it to an inequality of one variable only (of degree 3). Therefore, we can say that every cyclic inequality in three variables a, b, c of degree 3 is solvable.

∇

With the help of this theorem, we can prove many nice and hard cyclic inequalities of degree 3. Polynomial inequalities will be discussed first and the fraction inequalities will be mentioned in the end of this article.

1.10 Application for Polynomial Inequalities

Example 1.10.1. *Let a, b, c be non-negative real numbers. Prove that*

$$a^3 + b^3 + c^3 + \left(\frac{3}{\sqrt[3]{4}} - 1\right) abc \geq \frac{3}{\sqrt[3]{4}}(a^2b + b^2c + c^2a).$$

SOLUTION. Clearly, the above inequality is true if $a = b = c$. According to **CD3** theorem, it suffices to consider the inequality in one case $b = 1, c = 0$. The inequality becomes

$$a^3 + 1 \geq \frac{3}{\sqrt[3]{4}} a^2,$$

which simply follows from **AM-GM** inequality

$$a^3 + 1 = \frac{a^3}{2} + \frac{a^3}{2} + 1 \geq \frac{3}{\sqrt[3]{4}} a^2.$$

The equality holds for $a = b = c$ or $a = \sqrt[3]{2}, b = 1, c = 0$ up to permutation.

∇

Example 1.10.2. *Let a, b, c be non-negative real numbers with sum 3. Prove that*

$$a^2b + b^2c + c^2a + abc \leq 4.$$

SOLUTION. The inequality is equivalent to

$$27(a^2b + b^2c + c^2a + abc) \leq 4(a + b + c)^3.$$

Because this is a cyclic inequality and holds for $a = b = c$, we only need to consider the case $c = 0$ due to **CD3** theorem. In this case, the inequality becomes

$$24a^2b \leq 4(a + b)^3.$$

By **AM-GM** inequality, we have

$$(a + b)^3 = \left(\frac{a}{2} + \frac{a}{2} + b\right)^3 \geq \frac{27a^2b}{4}.$$

Therefore we are done. The equality holds for $a = b = c$ or $(a, b, c) \sim (2, 1, 0)$.

∇

Example 1.10.3. *Let a, b, c be non-negative real numbers. Prove that*

$$4(a^3 + b^3 + c^3) + 12(a^2b + b^2c + c^2a) \geq 15(ab^2 + bc^2 + ca^2) + 3abc.$$

SOLUTION. Because the inequality is cyclic and holds for $a = b = c$, according to **CD3** theorem, it is enough to consider the case $c = 0$. The inequality becomes

$$4(a^3 + b^3) + 12a^2b \geq 15ab^2,$$

or

$$(2a - b)^2(a + 4b) \geq 0,$$

which is obvious. The equality holds for $a = b = c$ or $(a, b, c) \sim (1, 2, 0)$.

∇

Example 1.10.4. *Let a, b, c be non-negative real numbers with sum 4. Prove that*

$$5(ab^2 + bc^2 + ca^2) + 3abc \leq 36 + 3(a^2b + b^2c + c^2a).$$

SOLUTION. Because the inequality is cyclic and holds for $a = b = c = \dfrac{4}{3}$, it suffices to consider it in case $c = 0$ and $a + b = 4$. We have to prove that

$$5ab^2 - 3a^2b \leq 36$$

or

$$a(4 - a)(5 - 2a) \leq 9.$$

Applying **AM-GM** inequality, we have the desired result

$$a(4 - a)(5 - 2a) = \frac{1}{3} \cdot (3a) \cdot (4 - a) \cdot (5 - 2a) \leq \frac{1}{81}(3a + 4 - a + 5 - 2a)^3 = 9.$$

The equality holds for $a = 1, b = 3, c = 0$ and every permutation.

∇

Example 1.10.5. *Let a, b, c be non-negative real numbers such that $a + b + c = 3$. For each $k \geq 0$, find the maximum value of*

$$a^2(kb + c) + b^2(kc + a) + c^2(ka + b).$$

(Pham Kim Hung)

SOLUTION. Because the expression is cyclic, we can assume first that $c = 0, a + b = 3$ and find the maximum value of

$$F = ka^2b + b^2a.$$

For $k = 1$, we have $F = ab(a + b) = 3ab \leq \dfrac{27}{4}$ by **AM-GM** inequality. Otherwise, assume that $k \neq 1$. We denote

$$f(a) = ka^2(3 - a) + a(3 - a)^2 = (1 - k)a^3 + 3(k - 2)a^2 + 9a.$$

We have

$$f'(a) = 3(1-k)a^2 + 6(k-2)a + 9.$$

Now we check the roots of the derivative and use that $a \in [0,3]$

$$f'(a) = 0 \;\Leftrightarrow\; a = a_0 = \frac{2-k-\sqrt{k^2-k+1}}{1-k} = \frac{\sqrt{k^2-k+1}+k-2}{k-1}.$$

The maximum of $f(a)$ is attained for $a = a_0$ and

$$\max_{a\in[0,3]} f(a) = f(a_0) = \frac{2(k^2-k+1)}{(k-1)^2}\left(\sqrt{k^2-k+1}+k-2\right) + \frac{3(k-2)}{k-1}.$$

According to **CD3** theorem, we conclude that the maximum of

$$a^2(kb+c) + b^2(kc+a) + c^2(ka+b)$$

is

$$\min\left\{3(k+1)\,;\; \frac{2(k^2-k+1)}{(k-1)^2}\left(\sqrt{k^2-k+1}+k-2\right) + \frac{3(k-2)}{k-1}\right\}.$$

Comment. According to this proof and **CD3** theorem, we also have the following similar result

★ *Given non-negative real numbers a,b,c such that $a+b+c=3$, for each $k \geq 0$, prove that*

$$a^2(kb+c) + b^2(kc+a) + c^2(ka+b) + mabc \geq$$

$$\min\left\{3(k+1)+m\,;\; \frac{2(k^2-k+1)}{(k-1)^2}\left(\sqrt{k^2-k+1}+k-2\right) + \frac{3(k-2)}{k-1}\right\}.$$

$$\nabla$$

Example 1.10.6. *Let a,b,c be non-negative real numbers such that $a+b+c=3$. For each $k \geq 0$, find the maximum and minimum value of*

$$a^2(kb-c) + b^2(kc-a) + c^2(ka-b).$$

(Pham Kim Hung)

SOLUTION. As in the preceding solutions, we will first consider the case $c=0$. Denote

$$ka^2b - ab^2 = ka^2(3-a) - a(3-a)^2 = -(k+1)a^3 + 3(k+2)a^2 - 9a = f(a),$$

then we get

$$f'(a) = -3(k+1)a^2 + 6(k+2)a - 9.$$

The equation $f'(a) = 0$ has exactly two positive real roots (in $[0,3]$)

$$a_1 = \frac{k+2-\sqrt{k^2+k+1}}{k+1}\;;\; a_2 = \frac{k+2+\sqrt{k^2+k+1}}{k+1}.$$

Therefore, we infer that

$$\min_{a\in[0,3]} f(a) = f(a_1) = \frac{2(k^2+k+1)}{(k+1)^2}\left(k+2-\sqrt{k^2+k+1}\right) - \frac{3(k+2)}{k+1} = m\,;$$

$$\max_{a\in[0,3]} f(a) = f(a_1) = \frac{2(k^2+k+1)}{(k+1)^2}\left(k+2+\sqrt{k^2+k+1}\right) - \frac{3(k+2)}{k+1} = M\,.$$

The inequality is cyclic, and therefore, due to **CD3** theorem, we conclude

$$\min\ a^2(kb-c)+b^2(kc-a)+c^2(ka-b) = \min\{3(k-1)\,;m\}\ ;$$

$$\max\ a^2(kb-c)+b^2(kc-a)+c^2(ka-b) = \max\{3(k-1)\,;M\}\ ;$$

Comment. According to this proof and **CD3** theorem, we obtain a similar result

★ *Given $a,b,c\ge 0$ such that $a+b+c=3$ and for each $k\ge 0$ and $m\in\mathbb{R}$, we have*

$$\min\{a^2(kb-c)+b^2(kc-a)+c^2(ka-b)+rabc\} = \min\{3(k-1)+r\,;m\}\ ;$$

$$\max\{a^2(kb-c)+b^2(kc-a)+c^2(ka-b)+rabc\} = \max\{3(k-1)+r\,;M\}\ .$$

$$\nabla$$

The following theorem is a generalization of the "cyclic inequality of degree 3" theorem for non-homogeneous inequalities.

Theorem 4 (CD3-improved)**.** *Let $P(a,b,c)$ be a cyclic polynomial of degree 3.*

$$P(a,b,c) = m\sum_{cyc}a^3 + n\sum_{cyc}a^2b + p\sum_{cyc}ab^2 + qabc + r\sum_{cyc}a^2 + s\sum_{cyc}ab + t\sum_{cyc}a + u.$$

The inequality $P\ge 0$ holds for all non-negative variables a,b,c if and only if

$$P(a,a,a)\ge 0\ ;\ P(a,b,0)\ge 0\ \forall a,b\ge 0;$$

PROOF. We fix the sum $a+b+c=A$ and prove that for all $A\ge 0$ then

$$Q(a,b,c) = m\sum_{cyc}a^3 + n\sum_{cyc}a^2b + p\sum_{cyc}ab^2 + qabc + \frac{r}{A}\left(\sum_{cyc}a^2 + s\sum_{cyc}ab\right)\left(\sum_{cyc}a\right)$$

$$+\frac{t}{A^2}\left(\sum_{cyc}a\right)^3 + \frac{u}{A^3}\left(\sum_{cyc}a\right)^3 \ge 0.$$

Since $Q(a,b,c)$ is a homogeneous and cyclic polynomial of degree 3, according to the **CD3** theorem, we can conclude that $Q(a,b,c)\ge 0$ if and only if

$$R(a) = Q(a,a,a)\ge 0\,; S(a,b) = Q(a,b,0)\ge 0\,.$$

Now, since both $R(a)$ and $S(a,b)$ are homogeneous polynomials we can normalize and prove that $R(A) \geq 0$ (assuming that $a = A$) and $Q(a,b) \geq 0$ (assuming that $a + b = A$). Since $S(A) = P(a,a,a) \geq 0$ and $S(a,b) = P(a,b,0) \geq 0$ by hypothesis, the theorem is proved completely.

∇

Here are some applications of this theorem.

Example 1.10.7. *Let a, b, c be non-negative real numbers. Prove that*

$$a^2 + b^2 + c^2 + 2 + \frac{4}{3}(a^2b + b^2c + c^2a) \geq 3(ab + bc + ca).$$

(Pham Kim Hung)

SOLUTION. For $a = b = c = t$, the inequality becomes

$$4t^3 - 6t^2 + 2 \geq 0 \;\Leftrightarrow\; 2(t-1)^2(2t+1) \geq 0.$$

This one is obvious, so we are done. Equality holds for $a = b = c = 1$.

For $c = 0$, the inequality becomes

$$\frac{4}{3}a^2b + a^2 + b^2 + 2 \geq 3ab$$

or

$$f(a) = \left(\frac{4}{3}b + 1\right)a^2 - 3b \cdot a + (b^2 + 2) \geq 0.$$

Since

$$\Delta_f = (3b)^2 - 4\left(\frac{4}{3}b + 1\right)(b^2 + 2) \leq 9b^2 - 4\,(b+1)\left(b^2 + 1\right) = -4b^3 + 5b^2 - 4b - 4 < 0,$$

the inequality is proved in this case as well. The conclusion follows immediately.

∇

Example 1.10.8. *Let a, b, c be non-negative real numbers. Prove that*

$$a^2 + b^2 + c^2 + 2(a^2b + b^2c + c^2a) + 12 \geq 6(a + b + c) + ab + bc + ca.$$

(Pham Kim Hung)

SOLUTION. For $a = b = c = t$, the inequality becomes $6t^3 + 12 \geq 18t$, or $6(t-1)^2(t+2) \geq 0$, which is obvious. Therefore it suffices to prove the inequality in case $c = 0$. In this case, we have to prove that

$$a^2 + b^2 + 2a^2b + 12 \geq 6(a + b)$$

or
$$f(a) = a^2(1+2b) - 6a + (b^2 - 6b + 12) \geq 0.$$
Since
$$\Delta'_f = 9 - (1+2b)(b^2 - 6b + 12) = -2b^3 + 11b^2 - 18b - 3 < 0,$$
so we are done. The equality holds for $a = b = c = 1$.

∇

Example 1.10.9. *Let a, b, c be non-negative real numbers. Prove that*
$$a^2 + b^2 + c^2 + 3 + \frac{1}{6}(a^2b + b^2c + c^2a + 15abc) \geq a + b + c + 2(ab + bc + ca).$$
(Pham Kim Hung)

SOLUTION. If $a = b = c$, the inequality is obvious. If $c = 0$, it becomes
$$a^2 + b^2 + 3 + \frac{a^2b}{6} \geq a + b + 2ab$$
or
$$a^2\left(1 + \frac{b}{6}\right) - (2b+1)a + (b^2 - b + 3) \geq 0.$$
It is easy to check that
$$\Delta = (2b+1)^2 - 4\left(1 + \frac{b}{6}\right)(b^2 - b + 3) = -\frac{2}{3}b^3 + \frac{2}{3}b^2 + 6b - 11.$$

Case $b \leq \frac{8}{5}$. We have
$$\Delta = -\frac{2}{3}\left(b^3 - b^2 - b + 1\right) - \left(\frac{31}{3} - \frac{16b}{3}\right) < 0.$$

Case $b \geq \frac{8}{5}$. We have
$$\Delta = -\left(\frac{1}{4}b^3 - 6b + 11\right) - \left(\frac{5}{12}b^3 - \frac{2}{3}b^2\right) < 0.$$
Therefore $\Delta < 0$ in every case, and we are done. Equality holds for $a = b = c = 1$.

∇

Example 1.10.10. *Let a, b, c be non-negative real numbers. Prove that*
$$2(a^3 + b^3 + c^3) + abc + ab + bc + ca + 2 \geq 2(a^2b + b^2c + c^2a) + a^2 + b^2 + c^2 + a + b + c.$$
(Pham Kim Hung)

SOLUTION. If $a = b = c$, the inequality is obvious, with equality for $a = b = c = 1$. Therefore we can assume that $c = 0$, and the inequality becomes

$$2(a^3 + b^3) + ab + 2 \geq 2a^2b + a^2 + b^2 + a + b.$$

We may assume that $a \geq b$. Denote

$$f(a) = 2(a^3 + b^3) + ab + 2 - 2a^2b - a^2 - b^2 - a - b.$$

If $a \geq 1$ then

$$f'(a) = 6a^2 - 4ab + b - 2a - 1 \geq 2a^2 - 3a + 1 = (a - 1)(2a - 1) \geq 0.$$

Therefore

$$f(a) \geq f(b) = 2b^3 - b^2 - 2b + 2 = 2(b - 1)^2(b + 1) + b^2 > 0.$$

Therefore we may assume that $a \leq 1$. It suffices to prove that

$$2(a^3 + b^3) + 2 \geq ab + a^2 + b^2 + a + b.$$

Let $x = a + b, y = ab$. The inequality becomes

$$2x(x^2 - 3y) + 2 \geq x^2 - y + x$$

or

$$2x^3 - x^2 - x + 2 + y(1 - 6x) \geq 0.$$

Since $0 \leq y \leq \dfrac{x^2}{4}$, the previous inequality is proved if we can prove that

$$2x^3 - x^2 - x + 2 \geq 0 \ \ (1)$$

$$2x^3 - x^2 - x + 2 + \frac{x^2}{4}(1 - 6x) \geq 0 \ \ (2)$$

Inequalities (1) and (2) can be proved easily, so we are done. The equality holds if and only if $a = b = c = 1$.

∇

The theorem *"Cyclic inequality of Degree 3"* is a natural generalization of Hoo Joo Lee's theorem from symmetry to cyclicity (for expressions of three variables). Similarly, we have a very nice generalization of Hoo Joo Lee's theorem for symmetric inequalities of 4 variables in degree 3. It is proved by the global derivative as well.

Proposition 1. *Let $P(a, b, c, d)$ be a symmetric polynomial of degree 3. The inequality $P \geq 0$ holds for all non-negative variables a, b, c, d if and only if*

$$P(1, 1, 1, 1) \geq 0\ ;\ P(1, 1, 1, 0) \geq 0\ ;\ P(1, 1, 0, 0) \geq 0\ ;\ P(1, 0, 0, 0) \geq 0\ 0$$

SOLUTION. The necessary condition is obvious. To prove the sufficient condition, we will use Hoo Joo Lee's theorem. WLOG, assume that

$$P(a,b,c,d) = \alpha\sum_{cyc} a^3 + \beta\sum_{cyc} a^2(b+c+d) + 6\gamma\sum_{cyc} abc.$$

Taking the derivative, we get the following polynomial

$$Q(a,b,c,d) = (3\alpha+3\beta)\sum_{cyc} a^2 + (2\beta+6\gamma)\sum_{cyc} a(b+c+d).$$

Clearly, $Q \geq 0$ because $\sum\limits_{cyc} a^2 \geq \frac{1}{3}\sum\limits_{cyc} a(b+c+d), 3\alpha+3\beta = \frac{3}{2}P(1,1,0,0) \geq 0$ and

$$(\alpha+3\beta)+3(2\beta+6\gamma) = 3(\alpha+3\beta+6\gamma) = \frac{3}{4}P(1,1,1,1) \geq 0.$$

Since $Q \geq 0$, according to the principle of the global derivative method and by the method of mixing all variables, it suffices to prove that

$$P(a,b,c,0) \geq 0.$$

Since $P(a,b,c,0)$ is a symmetric polynomial in a,b,c, we have the desired result due to Hoo Joo Lee's theorem.

∇

Proposition 2. *Let $P(x_1,x_2,...,x_n)$ be a third-degree symmetric polynomial*

$$P = \alpha\sum_{i=1}^{n} x_i^3 + \beta\sum_{i\neq j} x_i^2x_j + \gamma\sum_{i\neq j\neq k} x_ix_jx_k$$

such that $3\alpha+(n-1)\beta \geq 0$. The inequality $P(x_1,x_2,...,x_n) \geq 0$ holds for all non-negative variables $x_1,x_2,...,x_n$ if and only if

$$P(1,0,0,...,0) \geq 0\,;\ P(1,1,0,0,...,0) \geq 0\,;...;\ P(1,1,...,1,0) \geq 0\,;\ P(1,1,1,...,1) \geq 0\,.$$

PROOF. To prove this problem, we use induction. Assuming that the theorem is proved already for $n-1$ variables, we have to prove it for n variables. Generally, the polynomial $P(x_1,x_2,...,x_n)$ can be expressed in the following form

$$P = \alpha\sum_{i=1}^{n} x_i^3 + \beta\sum_{i\neq j} x_i^2x_j + \gamma\sum_{i\neq j\neq k} x_ix_jx_k.$$

Taking the global derivative, we get the polynomial

$$\begin{aligned}Q &= 3\alpha\sum_{i=1}^{n} x_i^2 + (n-1)\beta\sum_{i=1}^{n} x_i^2 + 2\beta\sum_{i\neq j} x_ix_j + 3(n-2)\gamma\sum_{i\neq j} x_ix_j \\ &= (3\alpha+(n-1)\beta)\sum_{i=1}^{n} x_i^2 + (2\beta+3(n-2)\gamma)\sum_{i\neq j} x_ix_j.\end{aligned}$$

Moreover, since $P(1,0,0,...,0) \geq 0, P(1,1,0,0,...,0) \geq 0$, we get

$$P(1,0,0,...,0) = \alpha \geq 0\ ;$$

Due to **AM-GM** inequality, we get

$$\sum_{i=1}^{n} x_i^2 \geq \frac{1}{n-1} \sum_{i \neq j} x_i x_j .$$

Since $P(1,1,1,...,1) \geq 0$, we get

$$P(1,1,...,1) = n\alpha + n(n-1)\beta + n(n-1)(n-2)\gamma \geq 0$$

$$\Rightarrow \alpha + (n-1)\beta + (n-1)(n-2)\gamma \geq 0$$

$$\Rightarrow \frac{3\alpha + (n-1)\beta}{n-1} + (2\beta + 3(n-2)\gamma) \geq 0.$$

Therefore

$$Q \geq \left(\frac{3\alpha + (n-1)\beta}{n-1} \right) \left(\sum_{i=1}^{n} x_i^2 - \sum_{i \neq j} x_i x_j \right).$$

By the principle of the global derivative and the mixing all variables method, we infer that, in order to prove $P(x_1, x_2, ..., x_n) \geq 0$, it suffices to prove $P(x_1, x_2, ..., x_{n-1}, 0) \geq 0$. Notice that $3\alpha + (n-1)\beta \geq 0\ ;\ \alpha \geq 0$, so $3\alpha + (n-2)\beta \geq 0$. The conclusion follows immediately by induction.

∇

Example 1.10.11. *Let a, b, c, d be non-negative real numbers. Prove that*

$$4(a^3 + b^3 + c^3 + d^3) + 15(abc + bcd + cda + dab) \geq (a+b+c+d)^3.$$

SOLUTION. Because this is a third-degree symmetric inequality of four variables, according to the generalization of the **SD3** theorem, it suffices to check this inequality in case $a = b = c = d = 1$ or $a = 0, b = c = d = 1$ or $a = b = 0, c = d = 1$ or $a = b = c = 0, d = 1$. They are all obvious, so we have the desired result. The equality holds for $a = b = c = d$ or $a = b = c, d = 0$ up to permutation.

∇

Example 1.10.12. *Let a, b, c, d be non-negative real numbers such that $a + b + c + d = 4$. Prove that*

$$a^3 + b^3 + c^3 + d^3 + 10(ab + bc + cd + da + ac + bd) \leq 64.$$

SOLUTION. The inequality can be rewritten as (homogeneous form)

$$a^3+b^3+c^3+d^3+\frac{5}{2}(a+b+c+d)(ab+bc+cd+da+ac+bd)\le(a+b+c+d)^3.$$

It is easy to check that the inequality holds for $(a,b,c,d)=(1,1,1,1)$ or $(1,1,1,0)$ or $(1,1,0,0)$ or $(1,0,0,0)$ (and it is an equality for $(1,1,1,1),(1,0,0,0)$). Therefore, the inequality is proved due to the previous proposition, with equality for $(1,1,1,1)$ and $(4,0,0,0)$.

∇

Example 1.10.13. *Let a,b,c,d be non-negative real numbers such that $a+b+c+d=4$. Prove that*

$$a^3+b^3+c^3+d^3+\frac{14}{3}(ab+bc+cd+da+ac+bd)\ge 32.$$

SOLUTION. The inequality can be rewritten as (homogeneous form)

$$a^3+b^3+c^3+d^3+\frac{7}{6}(a+b+c+d)(ab+bc+cd+da+ac+bd)\ge\frac{1}{2}(a+b+c+d)^3.$$

By the previous proposition/theorem, it suffices to consider this inequality in the cases $(a,b,c,d)\in\{(1,1,1,1),(1,1,1,0),(1,1,0,0),(1,0,0,0)\}$. In these cases, the inequality is clearly true, with equality for $(1,1,1,1)$ and $(1,1,1,0)$. Therefore the initial inequality is proved successfully, with equality for $(1,1,1,1)$ and $\left(\frac{4}{3},\frac{4}{3},\frac{4}{3},0\right)$.

∇

Example 1.10.14. *Let $a_1,a_2,...,a_n$ $(n\ge 3)$ be non-negative real numbers. Prove that*

$$\frac{n-1}{2}\sum_{i=1}^{n}a_i^3+\frac{3}{n-2}\sum_{cyc}a_1a_2a_3\ge\sum_{cyc}a_1a_2(a_1+a_2).$$

(Vasile Cirtoaje)

SOLUTION. In this problem, we have $\alpha=\dfrac{n-1}{2}$ and $\beta=-1$. Because

$$3\alpha+(n-1)\beta=\frac{3(n-1)}{2}-(n-1)=\frac{n-1}{2}>0,$$

according to the previous theorem (generalization for n variables), we get that it suffices to consider the initial inequality in case some of the variables $a_1,a_2,...,a_n$ are 1 and the other are 0. For $1\le k\le n$, assume that $a_1=a_2=...=a_k=1$ and $a_{k+1}=a_{k+2}=...=a_{k+n}=0$, then the inequality becomes

$$\frac{k(n-1)}{2}+\frac{3}{n-2}\cdot\frac{k(k-1)(k-2)}{6}\ge k(k-1)$$

$$\Leftrightarrow \; n-1+\frac{(k-1)(k-2)}{n-2} \geq 2(k-1)$$

$$\Leftrightarrow \; (n-k)(n-k+1) \geq 0.$$

This inequality is obvious because $k \in \{1, 2, ..., n\}$, and we are done. Equality holds for $a_1 = a_2 = ... = a_n$ or $a_1 = a_2 = ... = a_{n-1}, a_n = 0$.

∇

1.11 Applications for Fraction Inequalities

In this section, we will return to the theorem **CD3** presented before. Starting from **CD3** we will obtain the following result ("Cyclic inequalities of degree 3 for fractions"), a very general and useful result in proving fractional inequalities.

Proposition 3. *Consider the following expression of non-negative real numbers a, b, c*

$$F(a,b,c) = \frac{ma+nb+pc}{\alpha a+\beta b+\gamma c} + \frac{mb+nc+pa}{\alpha b+\beta c+\gamma a} + \frac{mc+na+pb}{\alpha c+\beta a+\gamma b},$$

in which $a, b, c \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}^+$. The inequality $F(a,b,c) \geq k$ holds for all $a, b, c \geq 0$ if and only if $F(1,1,1) \geq k$ and $F(a,b,0) \geq k$ for all $a, b \geq 0$.

PROOF. It is easy to see that this proposition is directly obtained from the theorem **CD3**. Indeed, consider the expression

$$G(a,b,c) = \sum_{cyc} (ma+bp+pc)\left(\alpha b+\beta c+\gamma a\right)\left(\alpha c+\beta a+\gamma b\right) - k \prod_{cyc} \left(\alpha a+\beta b+\gamma c\right).$$

By hypothesis, we have $G(1,1,1) \geq 0$ and $G(a,b,0) \geq 0$. Moreover, G is a cyclic polynomial of degree 3. The conclusion follows from the theorem **CD3** instantly.

$$\nabla$$

According to this propostion, we can make up a lot of beautiful and hard inequalities. Here are some of them.

Example 1.11.1. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{a+2b}{c+2b} + \frac{b+2c}{a+2c} + \frac{c+2a}{b+2a} \geq 3.$$

(Pham Kim Hung, Volume I)

SOLUTION. The inequality is cyclic and holds for $a = b = c$, so, according to the previous theorem, we can assume that $b = 1, c = 0$. In this case, we have to prove that

$$\frac{a+2}{2} + \frac{1}{a} + \frac{2a}{1+2a} \geq 3$$
$$\Leftrightarrow \ \frac{a}{2} + \frac{a+1}{a(1+2a)} \geq 1.$$

Applying **AM-GM** inequality for the left-hand expression, we get

$$\text{LHS} \geq \sqrt{\frac{2a(1+a)}{a(1+2a)}} \geq 1.$$

That means, the inequality holds for $c = 0$. The conclusion follows immediately.

$$\nabla$$

Example 1.11.2. *Let a, b, c be non-negative real numbers. For each $k \geq 0$, find the minimum of the expression*

$$\frac{a+kb}{c+kb}+\frac{b+kc}{a+kc}+\frac{c+ka}{b+ka}.$$

(Pham Kim Hung, Volume I)

SOLUTION. For $b = 1, c = 0$, the expression becomes

$$f(a)=\frac{a+k}{k}+\frac{1}{a}+\frac{ka}{1+ka}=2+\frac{a}{k}+\frac{1}{a}-\frac{1}{1+ka}.$$

If $k \leq 1$ then, according to **AM-GM** inequality, we have

$$f(a) \geq 2+\frac{2}{\sqrt{k}}-1 \geq 3.$$

We will now consider the main case, when $k \geq 1$. Clearly,

$$f(a)=2+\frac{a}{k}+\frac{1}{a}-\frac{1}{1+ka} \geq 2+\frac{a}{k}>2.$$

However, 2 is not the minimal value of $f(a)$. To find this value, we have to use derivatives. We have first

$$f'(a)=\frac{1}{k}-\frac{1}{a^2}+\frac{k}{(1+ka)^2}.$$

The equation $f'(a) = 0$ is equivalent to

$$k^2a^4+2ka^3-(k^3+k^2-1)a^2-2k^2a-k=0.$$

It is easy to check that the equation has exactly one real root a_0, so

$$\min f(a)=f(a_0)=2+\frac{a_0}{k}+\frac{1}{a_0}-\frac{1}{1+ka_0}.$$

According to the **CD3** theorem (for fractions), we conclude that

$$\min\left\{\frac{a+kb}{c+kb}+\frac{b+kc}{a+kc}+\frac{c+ka}{b+ka}\right\}=\min\left\{3\,;\;2+\frac{a_0}{k}+\frac{1}{a_0}-\frac{1}{1+ka_0}\right\}.$$

Comment. 1. Because $f(a) \geq 2\ \forall a > 0$, we conclude that

★ *For all non-negative real numbers a, b, c, k*

$$\frac{a+kb}{c+kb}+\frac{b+kc}{a+kc}+\frac{c+ka}{b+ka} \geq 2.$$

2. We can try some estimations for a_0. Clearly, $a_0 \leq \sqrt{k}$ and $a_0 \geq \sqrt{k-1}$, so

$$2+\frac{a_0}{k}+\frac{1}{a_0}-\frac{1}{1+ka_0} \geq \frac{2}{\sqrt{k}}-\frac{1}{1+k\sqrt{k-1}},$$

$$\Rightarrow \min\left\{\frac{a+kb}{c+kb}+\frac{b+kc}{a+kc}+\frac{c+ka}{b+ka}\right\}=\min\left\{3\,;2+\frac{2}{\sqrt{k}}-\frac{1}{1+k\sqrt{k-1}}\right\}\ \forall k>1,$$

and we get a very nice result

★ *For all non-negative real numbers a, b, c*

$$\frac{a+\sqrt{7}b}{c+\sqrt{7}b}+\frac{b+\sqrt{7}c}{a+\sqrt{7}c}+\frac{c+\sqrt{7}a}{b+\sqrt{7}a}\ge 3.$$

∇

Example 1.11.3. *Let a, b, c be non-negative real numbers. Prove that*

$$1\le\frac{a+b}{a+4b+c}+\frac{b+c}{b+4c+a}+\frac{c+a}{c+4a+b}\le\frac{4}{3}.$$

(Pham Kim Hung)

SOLUTION. For $b=1, c=0$, the inequality becomes

$$\frac{a+1}{a+4}+\frac{1}{a+1}+\frac{a}{4a+1}\ge 1\ (\star)$$

and

$$\frac{a+1}{a+4}+\frac{1}{a+1}+\frac{a}{4a+1}\le\frac{4}{3}\ (\star\star)$$

The inequality $(\star)$ is equivalent to (after expanding)

$$a^3-3a^2+7a+5\ge 0,$$

which is obvious because

$$a^3-3a^2+7a^2\ge\left(2\sqrt{7}-3\right)a^2\ge 0.$$

The inequality $(\star\star)$ is equivalent to (after expanding)

$$a^3+20a^2-3a+1\ge 0,$$

which is obvious, too, because

$$20a^2-3a+1\ge\left(2\sqrt{20}-3\right)a\ge 0.$$

Because the inequality holds if one of a, b, c is equal to 0, it is also true if $a=b=c$. Therefore, we have the conclusion due to the "Cyclic inequalities of degree 3- theorem for fractions". Only the left-hand inequality has an equality case for $a=b=c$.

∇

Example 1.11.4. *Let a, b, c be non-negative real numbers. Prove that*

$$\sqrt{\frac{a}{4a+4b+c}}+\sqrt{\frac{b}{4b+4c+a}}+\sqrt{\frac{c}{4c+4a+b}} \le 1.$$

(Pham Kim Hung, Volume I)

SOLUTION. It suffices to prove that

$$\frac{a}{4a+4b+c}+\frac{b}{4b+4c+a}+\frac{c}{4c+4a+b} \le \frac{1}{3} \quad (\star)$$

For $c = 0$, the inequality becomes

$$\frac{a}{4a+4b}+\frac{b}{4b+a} \le \frac{1}{3}$$

or (after expanding)

$$(a-2b)^2 \ge 0.$$

Therefore we are done according to the proposition. In $(\star)$, the equality holds for $a = b = c$ or $(a, b, c) \sim (2, 1, 0)$. In the initial inequality, the equality only holds for $a = b = c$.

∇

Example 1.11.5. *Let a, b, c be non-negative real numbers. For each $k, l \ge 0$, find the maximal and minimal value of the expression*

$$\frac{a}{ka+lb+c}+\frac{b}{kb+lc+a}+\frac{c}{kc+la+b}.$$

(Pham Kim Hung)

SOLUTION. First we will examine the expression in case $b = 1, c = 0$. Denote

$$f(a) = \frac{a}{ka+l}+\frac{1}{k+a},$$

then we get

$$f'(a) = \frac{l}{(ka+l)^2}-\frac{1}{(k+a)^2}.$$

The equation $f'(a) = 0$ has exactly one positive real root $a = \sqrt{l}$. So, if $l > k^2$ then

$$\min\ f(a) = f\left(\sqrt{l}\right) = \frac{2}{k+\sqrt{l}}\ ;$$

$$\sup f(a) = \lim_{a\to+\infty} f(a) = \frac{1}{k}\ .$$

Otherwise, if $l < k^2$ then

$$\min\ f(a) = f(0) = \frac{1}{k}\ ;$$

$$\max\ f(a) = f\left(\sqrt{l}\right) = \frac{2}{k+\sqrt{l}}\ .$$

For $l = k^2$ then we have

$$f(a) = \frac{a}{ka+k^2} + \frac{1}{k+a} = \frac{1}{k}\ .$$

Denote by

$$F = \frac{a}{ka+lb+c} + \frac{b}{kb+lc+a} + \frac{c}{kc+la+b}.$$

According to the previous results, we have the conclusion

$$\min F = \min\left\{\frac{3}{k+l+1}\ ;\ \frac{1}{k}\ ;\ \frac{2}{k+\sqrt{l}}\right\}\ ;$$

$$\sup F = \max\left\{\frac{3}{k+l+1}\ ;\ \frac{1}{k}\ ;\ \frac{2}{k+\sqrt{l}}\right\}\ ;$$

Comment. According to this general result, we can write down a lot of nice inequalities such as

★ *Given non-negative real numbers a, b, c and for all $k \geq 1$, prove that*

$$\frac{3}{k^2+k+1} \leq \frac{a}{ka+k^2b+c} + \frac{b}{kb+k^2c+a} + \frac{c}{kc+k^2a+b} \leq \frac{1}{k}.$$

★ *Given non-negative real numbers a, b, c, and for all $k \geq 0$, prove that*

$$\frac{a}{ka+(2k-1)b+c} + \frac{b}{kb+(2k-1)c+a} + \frac{c}{kc+(2k-1)a+b} \geq \frac{1}{k}.$$

★ *Given non-negative real numbers a, b, c, and for all $k \geq 0$, prove that*

$$\frac{(k^2-k+1)a}{(2k^2-3k+2)a+k^2b+c} + \frac{(k^2-k+1)b}{(2k^2-3k+2)b+k^2c+a} + \frac{(k^2-k+1)c}{(2k^2-3k+2)c+k^2a+b} \leq 1.$$

In the last inequality, by letting $k = 3$, we obtain the following result

★ *Given non-negative real numbers a, b, c, prove that*

$$\frac{a}{11a+9b+c} + \frac{b}{11b+9c+a} + \frac{c}{11c+9a+b} \leq \frac{1}{7}.$$

Notice that the equality holds for $a = b = c$ or $(a, b, c) \sim (3, 1, 0)$.

∇

Example 1.11.6. *Let a, b, c be non-negative real numbers. Prove that*

$$\frac{3a+2b+c}{a+2b+3c} + \frac{3b+2c+a}{b+2c+3a} + \frac{3c+2a+b}{c+2a+3b} \geq 3.$$

SOLUTION. First we have to consider the inequality in case $b = 1, c = 0$. In this case, the inequality becomes

$$\frac{3a+2}{a+2} + \frac{3+a}{3a+1} + \frac{2a+1}{2a+3} \geq 3$$
$$\Leftrightarrow \frac{2a}{a+2} + \frac{3+a}{3a+1} \geq \frac{2a+5}{2a+3},$$

or (after directly expanding)

$$4a^3 + 3a^2 - 3a + 4 \geq 0.$$

This last inequality is obvious by **AM-GM** inequality

$$3a^2 - 3a + 4 \geq \left(2\sqrt{12} - 3\right) a \geq 0.$$

The inequality is true in case $abc = 0$. It is also obvious if $a = b = c$. According to the main theorem, we have the desired result.

∇

Example 1.11.7. *Let a, b, c be non-negative real numbers with sum 3. Prove that*

$$\frac{a+b}{1+b} + \frac{b+c}{1+c} + \frac{c+a}{1+a} \geq 3.$$

SOLUTION. The inequality is equivalent to (homogeneous form)

$$\frac{a+b}{b + \frac{a+b+c}{3}} + \frac{b+c}{c + \frac{a+b+c}{3}} + \frac{c+a}{a + \frac{a+b+c}{3}} \geq 3.$$

Because this problem is cyclic, homogeneous and holds if $a = b = c$, we can assume that $c = 0$ and $a + b = 3$. In this case, we have to prove that

$$\frac{3}{1+b} + b + \frac{a}{1+a} \geq 3$$
$$\Leftrightarrow \frac{3}{4-a} + \frac{a}{1+a} \geq a$$
$$\Leftrightarrow a^3 - 4a^2 + 3a + 3 \geq 0$$
$$\Leftrightarrow a(a-2)^2 + (3-a) \geq 0$$

which is obvious because $a \in [0, 3]$. The equality holds for only one case, $a = b = c = 1$.

∇

Article 5

Integral and Integrated Inequalites

1.12 Getting started

As a matter of fact, using integrals in proving inequalities is a new preoccupation in elementary mathematics. Although integrals are mainly used in superior mathematics, only the following results are used in this article

1. If $f(x) \geq 0 \ \forall x \in [a,b]$ then $\int_a^b f(x)dx \geq 0$.
2. If $f(x) \geq g(x) \ \forall x \in [a,b]$ then $\int_a^b f(x)dx \geq \int_a^b g(x)dx$.

Example 1.12.1. *Let $a_1, a_2, ..., a_n$ be real numbers. Prove that*

$$\sum_{i,j=1}^{n} \frac{a_i a_j}{i+j} \geq 0.$$

(Romania MO)

SOLUTION. This problem shows the great advantage of the integral method because other solutions are almost impossible. Indeed, consider the following function

$$f(x) = \sum_{i,j=1}^{n} a_i a_j x^{i+j-1} = \frac{1}{x}\left(\sum_{i,j=1}^{n} a_i x^i\right)^2.$$

So we have $f(x) \geq 0$ for all $x \geq 0$, therefore $\int_0^1 f(x)dx \geq 0$. Notice that

$$\int_0^1 f(x)dx = \sum_{i,j=1}^{n} \frac{a_i a_j}{i+j},$$

so the desired result follows immediately. We are done.

∇

Example 1.12.2. *Let a, b, c be positive real numbers such that $a+b+c=1$. Prove that*

$$(ab+bc+ca)\left(\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a}\right) \geq \frac{3}{4}.$$

(Gabriel Dospinescu)

SOLUTION. We will prove first the following result: for all $x \geq 0$

$$\frac{a}{(x+b)^2} + \frac{b}{(x+c)^2} + \frac{c}{(x+a)^2} \geq \frac{1}{(x+ab+bc+ca)^2} \quad (\star)$$

Indeed, by **Cauchy-Schwarz** inequality, we obtain

$$\left(\sum_{cyc} \frac{a}{(x+b)^2}\right)\left(\sum_{cyc} a\right) \geq \left(\sum_{cyc} \frac{a}{x+b}\right)^2,$$

or equivalently (because $a+b+c=1$)

$$\sum_{cyc} \frac{a}{(x+b)^2} \geq \left(\sum_{cyc} \frac{a}{x+b}\right)^2 \quad (1)$$

Applying **Cauchy-Schwarz** inequality again

$$\left(\sum_{cyc} \frac{a}{x+b}\right)\left(\sum_{cyc} a(x+b)\right) \geq (a+b+c)^2,$$

or equivalently

$$\sum_{cyc} \frac{a}{x+b} \geq \frac{1}{x+ab+bc+ca} \quad (2)$$

Results (1) and (2) combined show $(\star)$ immediately. According to $(\star)$, we have

$$\int_0^1 \left(\frac{a}{(x+b)^2} + \frac{b}{(x+c)^2} + \frac{c}{(x+a)^2}\right) \geq \int_0^1 \left(\frac{1}{(x+ab+bc+ca)^2}\right),$$

or

$$\frac{a}{b^2+b} + \frac{b}{c^2+c} + \frac{c}{a^2+a} \geq \frac{1}{(ab+bc+ca)(1+ab+bc+ca)},$$

and the problem is solved by the simple observation that $ab+bc+ca \leq \frac{1}{3}$.

∇

Example 1.12.3. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{b(1+ab)} + \frac{1}{c(1+bc)} + \frac{1}{a(1+ca)} \geq \frac{1}{\sqrt[3]{abc}\left(1+\sqrt[3]{a^2b^2c^2}\right)} \quad (1)$$

(Pham Kim Hung)

SOLUTION. Taking into account example **??**, we have

$$\frac{a}{(x+ab)^2}+\frac{b}{(x+bc)^2}+\frac{c}{(x+ca)^2}\geq\frac{3\sqrt[3]{abc}}{\left(x+\sqrt[3]{a^2b^2c^2}\right)^2}.$$

Integrating on $[0,1]$, we get

$$\sum_{cyc}\int_0^1\frac{a}{(x+ab)^2}\geq\int_0^1\frac{3\sqrt[3]{abc}}{\left(x+\sqrt[3]{a^2b^2c^2}\right)^2}$$

or

$$\frac{1}{b(1+ab)}+\frac{1}{c(1+bc)}+\frac{1}{a(1+ca)}\geq\frac{3}{\sqrt[3]{abc}\left(1+\sqrt[3]{a^2b^2c^2}\right)}.$$

Comment. 1. By integrating on $[0,abc]$, we obtain the following result

★ *Let a,b,c be positive real numbers. Prove that*

$$\frac{a}{b(1+bc)}+\frac{b}{c(1+bc)}+\frac{c}{a(1+ab)}\geq\frac{3}{1+\sqrt[3]{abc}} \tag{2}$$

2. By integrating on $[0,\sqrt[3]{abc}]$, we obtain the following result

★ *Let a,b,c be positive real numbers. Prove that*

$$\frac{1}{b(\sqrt[3]{abc}+bc)}+\frac{1}{c(\sqrt[3]{abc}+bc)}+\frac{1}{a(\sqrt[3]{abc}+ab)}\geq\frac{3}{abc+\sqrt[3]{a^2b^2c^2}} \tag{3}$$

3. Letting $abc=1$ in each of these inequalities, we obtain the following result

★ *Let a,b,c be positive real numbers such that $abc=1$. Prove that*

$$\frac{1}{a(1+ab)}+\frac{1}{b(1+bc)}+\frac{1}{c(1+ca)}\geq\frac{3}{2} \tag{4}$$

4. These inequalities can be solved by letting $a=\dfrac{kx}{y}, b=\dfrac{ky}{z}, c=\dfrac{kz}{x}$.

1.13 Integrated Inequalities

The main idea of this method and is hidden in the following example proposed by Gabriel Dospinescu in Mathlinks Forum.

Example 1.13.1. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{3}{3a+b}+\frac{3}{3b+c}+\frac{3}{3c+a}.$$

(Gabriel Dospinescu)

SOLUTION. There seems to be no purely algebraic solution to this hard inequality, however, the integral method makes up a very impressive one.

According to a well-known inequality (see problem **??** in volume I), we have

$$(x^2+y^2+z^2)^2\geq 3(x^3y+y^3z+z^3x).$$

Denote $x=t^a, y=t^b, z=t^c$, then this inequality becomes

$$\left(t^{2a}+t^{2b}+t^{2c}\right)^2\geq 3\left(t^{3a+b}+t^{3b+c}+t^{3c+a}\right).$$

Integrating both sides on $[0,1]$, we deduce that

$$\int_0^1\frac{1}{t}\left(t^{2a}+t^{2b}+t^{2c}\right)^2dt\geq 3\int_0^1\frac{1}{t}\left(t^{3a+b}+t^{3b+c}+t^{3c+a}\right)dt,$$

and the desired result follows immediately

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a}\geq\frac{3}{3a+b}+\frac{3}{3b+c}+\frac{3}{3c+a}.$$

We are done. The equality occurs if and only if $a=b=c$.

∇

This is a very ingenious and unexpected solution! The idea of replacing x, y, z with $x=t^a, y=t^b, z=t^c$ changes the initial inequality completely! This method can produce a lot of beautiful inequalities such as the ones that will be shown right now.

Example 1.13.2. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{3a}+\frac{1}{3b}+\frac{1}{3c}+\frac{3}{a+b+c}\geq\frac{1}{2a+b}+\frac{1}{2b+c}+\frac{1}{2c+a}+\frac{1}{2b+a}+\frac{1}{2c+b}+\frac{1}{2a+c}.$$

SOLUTION. Starting from **Schur** inequality, we have

$$x^3+y^3+z^3+3xyz\geq xy(x+y)+yz(y+z)+zx(z+x).$$

Letting now $x = t^a, y = t^b, z = t^c$, the above inequality becomes

$$\frac{1}{t}\left(t^{3a} + t^{3b} + t^{3c} + 3t^{a+b+c}\right) \geq \frac{1}{t}\left(t^{a+b}(t^a + t^b) + t^{b+c}(t^b + t^c) + t^{c+a}(t^c + t^a)\right).$$

Integrating both sides on $[0, 1]$, we have the desired result immediately.

∇

Example 1.13.3. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{5}{3a+b} + \frac{5}{3b+c} + \frac{5}{3c+a} \geq \frac{9}{a+3b} + \frac{9}{b+3c} + \frac{9}{c+3a}.$$

(Pham Kim Hung)

SOLUTION. Let $k = \dfrac{5}{4}$. We start from the following inequality

$$x^4 + y^4 + z^4 + k(x^3y + y^3z + z^3x) \geq (1 + \sqrt{2})(xy^3 + yz^3 + zx^3) \qquad (1)$$

Denoting $x = t^a, y = t^b, z = t^c$, we obtain

$$\sum_{cyc} t^{4a-1} + k\sum_{cyc} t^{3a+b-1} \geq (1+k)\sum_{cyc} t^{a+3b-1}.$$

Integrating both sides on $[0, 1]$, we get the desired result.

Finally, we have to prove the inequality (1) with the help of the global derivative and the mixing all variables method. Taking the global derivative, it becomes

$$4\sum_{cyc} x^3 + k\left(3\sum_{cyc} x^2y + \sum_{cyc} x^3\right) \geq (1+k)\left(\sum_{cyc} y^3 + 3\sum_{cyc} xy^2\right)$$

or

$$4\sum_{cyc} x^3 + 3k\sum_{cyc} x^2y \geq 3\,(1+k)\sum_{cyc} xy^2.$$

Since this is a cyclic inequality of degree 3, we may assume that $z = 0$ and we should prove next that

$$4(x^3 + y^3) + 3kx^2y \geq 3\,(1+k)\,xy^2.$$

This follows from **AM-GM** inequality immediately since

$$4y^3 + 3kx^2y \geq 4\sqrt{3k}xy^2 \geq 3\,(1+k)\,xy^2.$$

Then we only need to prove (1) in case $y = 1, z = 0$. In this case, (1) becomes

$$x^4 + 1 + kx^3 \geq (1+k)\,x.$$

Denote $f(x) = x^4 + kx^3 - (1+k)\,x + 1$, then we get

$$f'(x) = 4x^3 + 3kx^2 - (1+k)\,.$$

This function has exactly one positive real root $x = x_0 \approx 0.60406$, therefore

$$f(x) \geq f(x_0) \approx 0.05 > 0.$$

∇

Example 1.13.4. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{8}{a+b+c} \geq \frac{17}{3}\left(\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a}\right).$$

(Pham Kim Hung)

SOLUTION. Let $k = \dfrac{8}{9}$, then we need to prove that

$$\frac{1}{3a} + \frac{1}{3b} + \frac{1}{3c} + \frac{3k}{a+b+c} \geq (k+1)\left(\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a}\right).$$

According to example 1.10.1, we have

$$x^3 + y^3 + z^3 + 3kxyz \geq (k+1)\left(x^2y + y^2z + z^2x\right) \ (\star)$$

Denote $x = t^a, y = t^b, z = t^c$, then the previous inequality is transformed to

$$t^{3a-1} + t^{3b-1} + t^{3c-1} + 3kt^{a+b+c-1} \geq (k+1)\left(t^{2a+b-1} + t^{2b+c-1} + t^{c+a-1}\right).$$

We conclude that

$$\int_0^1 \left(t^{3a-1} + t^{3b-1} + t^{3c-1} + 3kt^{a+b+c-1}\right) dt \geq (k+1)\int_0^1 \left(t^{2a+b-1} + t^{2b+c-1} + t^{c+a-1}\right) dt$$

$$\Rightarrow \frac{1}{3a} + \frac{1}{3b} + \frac{1}{3c} + \frac{8}{3(a+b+c)} \geq \frac{17}{9}\left(\frac{1}{2a+b} + \frac{1}{2b+c} + \frac{1}{2c+a}\right),$$

which is exactly the desired result. The equality holds for $a = b = c$.

∇

According to these examples and their solutions, we realize that each fractional inequality has another corresponding inequality. They stand in couples - primary and integrated inequalities - if the primary one is true, the integrated one is true, too (but not vice versa). Yet if you still want to discover more about this interesting relationship, the following examples should be helpful.

Example 1.13.5. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{3}{a+b+c} \geq \frac{4}{5a+b}+\frac{4}{5b+c}+\frac{4}{5c+a}+\frac{4}{5b+a}+\frac{4}{5c+b}+\frac{4}{5a+c}.$$

SOLUTION. We can construct it from the primary inequality

$$a^6+b^6+c^6+a^2b^2c^2 \geq \frac{2}{3}\left(a^5(b+c)+b^5(c+a)+c^5(a+b)\right).$$

∇

Example 1.13.6. *Let a, b, c, d be positive real numbers with sum 4. Prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} \geq \frac{8}{(a+b)(c+d)}+\frac{8}{(b+c)(d+a)}+\frac{8}{(c+a)(b+d)}-\frac{8}{a+b+c+d}.$$

SOLUTION. It is easy to realize that this result is obtained from Turkevici's inequality

$$a^4+b^4+c^4+d^4+2abcd \geq a^2b^2+b^2c^2+c^2d^2+d^2a^2+a^2c^2+b^2d^2.$$

∇

Example 1.13.7. *Let a, b, c be positive real numbers. Prove that*

$$\frac{1}{4a}+\frac{1}{4b}+\frac{1}{4c}+\frac{1}{3a+b}+\frac{1}{3b+c}+\frac{1}{3c+a} \geq \frac{2}{a+3b}+\frac{2}{b+3c}+\frac{2}{c+3a}.$$

(Vasile Cirtoaje)

SOLUTION. We use the following familiar result (shown in the first volume)

$$a^4+b^4+c^4+a^3b+b^3c+c^3a \geq 2(ab^3+bc^3+ca^3)$$

to deduce the desired result. The equality holds for $a=b=c$.

∇

Example 1.13.8. *Let a, b, c be positive real numbers. Prove that*

$$\frac{4}{a}+\frac{4}{b}+\frac{4}{c}+\frac{36}{2a+b}+\frac{36}{2b+c}+\frac{36}{2c+a} \geq \frac{45}{a+2b}+\frac{45}{b+2c}+\frac{45}{c+2a}+\frac{9}{a+b+c}.$$

SOLUTION. We use the following familiar result

$$27(ab^2+bc^2+ca^2+abc) \leq 4(a+b+c)^3$$

to deduce the desired result. Equality holds for only $a=b=c$.

∇

Article 6

Two Improvements of the Mixing Variables Method

1.14 Mixing Variables by Convex Functions

Using convex functions is a very well-known approach in proving inequalities, so it would really be a mistake not to discuss them here. First, I want to return to a very familiar inequality from the previous chapter, which can be proved in six different ways with **SOS** method, **SMV** theorem, **UMV** theorem, the global derivative and also using the general induction method and the n**SMV** theorem (in the next section). We are talking about Turkevici's inequality

Example 1.14.1. *Let a, b, c, d be non-negative real numbers. Prove that*

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2.$$

Analysis. As a matter of fact, **SMV** theorem gives a "one-minute" solution; however, we sometimes don't want to use **SMV** theorem in the proof (suppose you are having a Mathematics Contest, then you can not re-build the general-mixing-variable lemma then the **SMV** theorem again. In this section I will help you handle this matter. Let us review some problems from Volume I. Reading thoroughly their solutions may bring a lot of interesting things in your mind.

Example 1.14.2. *Let a, b, c, d be positive real numbers with sum 4. Prove that*

$$abc + bcd + cda + dab + a^2b^2c^2 + b^2c^2d^2 + c^2d^2a^2 + d^2a^2b^2 \leq 8.$$

(Phan Thanh Nam, Volume I)

Example 1.14.3. *Prove that $a^4 + b^4 + c^4 + d^4 \geq 28abcd$ for all $a, b, c, d > 0$ satisfying*

$$(a + b + c + d)^2 = 3(a^2 + b^2 + c^2 + d^2).$$

(Pham Kim Hung, Volume I)

Analysis. Generalizing this solution, we can figure out a very simple and useful method of proving four variable inequalities. The common method we use is to rewrite the inequality by replacing $s_1 = a + b, x_1 = ab$ and $s_2 = c + d, x_2 = cd$. Regarding it as a function of $x_1 = ab$ and $x_2 = cd$, we can show that the expression attains the maximum or minimum when $a = b, c = d$ or $abcd = 0$. Let us see a detailed solution.

SOLUTION. (for Turkevici's inequality) Denote $m = a^2 + b^2, n = c^2 + d^2$ and $x = ab, y = cd$. We can rewrite the inequality in the following form

$$m^2 - 2x^2 + n^2 - 2y^2 + 2xy \geq x^2 + y^2 + (m^2 - 2x)(n^2 - 2y)$$

or

$$f(x, y) = -2x^2 - 2y^2 + 2xy + m^2 + n^2 - (m^2 - 2x)(n^2 - 2y) \geq 0.$$

Let us imagine that m and n are fixed as two constants. The variables x and y can vary freely but $2x \leq m$ and $2y \leq n$. Since the function f, considered as a one-variable function in each variable x, y, is concave (the coefficients of both x^2 and y^2 are -1), we get that

$$\min f(x, y) = \min f(\alpha, \beta)$$

where $\alpha \in \left\{0, \frac{m}{2}\right\}$ and $\beta \in \left\{0, \frac{m}{2}\right\}$ (we use the proposition that if a n-variable function is concave when we consider it as a one-variable function of each of its initial variables, then the minimum of this function is attained at its boundaries - this is one of the propositions on convex function in Volume I). Now if $\alpha = \frac{m}{2}$ and $\beta = \frac{n}{2}$ then we may assume that $a = b, c = d$. In this case, the inequality becomes obvious

$$2(a^4 + c^4) + 2a^2c^2 \geq a^4 + b^4 + 4a^2c^2 \;\Leftrightarrow\; (a^2 - c^2)^2 \geq 0.$$

Otherwise, we must have $\alpha\beta = 0$. In other words, we may assume that $abcd = 0$. WLOG, assume that $d = 0$, then the inequality becomes

$$a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2,$$

which is obvious, too. Therefore we are done in every case, and the desired result follows.

∇

Analysis. In this solution, we exploit the relationships $0 \leq x \leq \frac{m}{2}, 0 \leq y \leq \frac{n}{2}$. Moreover, we should notice that it is necessary to fix m and n first and let x, y vary (we can do that because for all $0 \leq x_0 \leq \frac{m}{2}$, there exist two numbers a, b such that $ab = m$ and $a^2 + b^2 = n^2$). For further analysis , see the solution to another familiar inequality already solved by **SMV** theorem.

Example 1.14.4. *Let a, b, c, d be non-negative positive real numbers with sum 1. Prove that*

$$abc + bcd + cda + dab \le \frac{1}{27} + \frac{176}{27}abcd.$$

SOLUTION. In this problem, we fix $a + b = m$ and $c + d = n$. Let $x = ab$ and $y = cd$ then

$$abc + bcd + cda + dab - \frac{1}{27} - \frac{176}{27}abcd = my + nx - \frac{1}{27} - \frac{176xy}{27} = f(x, y)$$

is a linear (convex) function in both x and y. It only reaches the maximum at boundary values, namely

$$\max f(x, y) = f(\alpha, \beta)\ ; \alpha \in \left\{0, \frac{m^2}{4}\right\}\ ;\ \beta \in \left\{0, \frac{n^2}{4}\right\}\ .$$

If $\alpha = \dfrac{m^2}{4}$ and $\beta = \dfrac{n^2}{4}$, we have $a = b, c = d$. In this case, the problem becomes

$$2(a^2c + c^2a) \le \frac{1}{27} + \frac{176a^2c^2}{27},$$

for all non-negative real numbers a, c and $a + c = \dfrac{1}{2}$. This inequality is equivalent to

$$176a^2c^2 + 1 \ge 27ac,$$

which is obviously true since $ac \le \dfrac{1}{16}$. The equality holds for $a = b = c = d = \dfrac{1}{4}$. Otherwise, if $\alpha \ne \dfrac{m^2}{4}$ and $\beta \ne \dfrac{n^2}{4}$, we must have $mn = 0$ or $abcd = 0$. WLOG, assume that $d = 0$, the inequality becomes $abc \le \dfrac{1}{27}$ if $a + b + c = 1$. This follows immediately from **AM-GM** inequality and attains equality for $a = b = c = \dfrac{1}{3}$. We are done.

∇

These solutions suggest two ways of using this techique: the first way is to fix $a + b, c + d$ and the second way is to fix $a^2 + b^2, c^2 + d^2$. Sometimes, we may fix $a^2 + b^2, c^2 + d^2$ and consider $a + b, c + d$ as variables, etc. All these are directed towards the same objective to make $a = b, c = d$ or $abcd = 0$. This is the reason why I consider this as a mixing variable method (to make $a = b$ or $ab = 0$). Finally, we have some applications of this very simple and elementary technique.

Example 1.14.5. *Let a, b, c, d be non-negative real numbers with sum 4. Prove that*

$$a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2 + 10abcd \le 16.$$

(Pham Kim Hung)

SOLUTION. We fix $a+b=m$ and $c+d=n$. Let $ab=x$ and $cd=y$ then

$$\sum_{cyc} a^2b^2 + 10abcd = x^2 + y^2 + (m^2-2x)(n^2-2y) + 10xy$$

are convex functions in each variable x and y. Therefore we only need to consider the case

$$x \in \left\{0\,;\ \frac{m^2}{4}\right\}\ ;\ y \in \left\{0\,;\ \frac{n^2}{4}\right\}\ .$$

Consider the first case $x = \dfrac{m^2}{4}$ and $y = \dfrac{n^2}{4}$. In this case, we have $a=b$ and $c=d$. The inequality becomes

$$a^4 + c^4 + 14a^2c^2 \le (a+c)^4.$$

Clearing similar teams, we get the following inequality

$$4(a^3c + c^3a) \ge 8a^2c^2,$$

which follows from **AM-GM** inequality. The equality holds for $a=b=c=d=1$. Now it's time for the second case $xy=0$ or $abcd=0$. WLOG, assume that $d=0$. The inequality becomes

$$a^2b^2 + b^2c^2 + c^2a^2 \le 16.$$

WLOG, assume that $a \ge b \ge c$. Since $a+b+c=4$, we infer that

$$a^2b^2 + b^2c^2 + c^2a^2 \le a^2b^2 + 2a^2bc \le a^2(b+c)^2 \le 16.$$

This ends the proof. The equality holds for $a=b=c=d=1$ or $a=b=2, c=d=0$ or permutations.

∇

Example 1.14.6. *Let a,b,c,d be non-negative real numbers with sum 4. Prove that*

$$(1+3a)(1+3b)(1+3c)(1+3d) \le 125 + 131abcd$$

(Pham Kim Hung)

SOLUTION. We fix $a+b=m$ and $c+d=n$. Let $x=ab$ and $y=cd$ (we regard x and y as variables). The inequality becomes

$$(1+9y+3n)(1+9x+3m) - 125 - 131xy \ge 0.$$

This expression is a linear (and also convex) function in each variable x and y, we get that it suffices to consider the case

$$x \in \left\{0\,;\ \frac{m^2}{4}\right\}\ ;\ y \in \left\{0\,;\ \frac{n^2}{4}\right\}\ ;$$

If $xy = abcd = 0$, we infer that one of the four numbers a, b, c, d is 0 and the inequality becomes obvious. Otherwise, we must have $x = \dfrac{m^2}{4}$ and $y = \dfrac{n^2}{4}$, or in other words, $a = b$ and $c = d$. The condition becomes $a + c = 2$ and the inequality that remains is

$$(1 + 3a)^2(1 + 3c)^2 \le 125 + 131a^2c^2$$

or

$$(7 + 9ac)^2 \le 125 + 131ac.$$

This inequality is obvious since $ac \le 1$. This ends the proof.

∇

I believe that these examples are enough for you to comprehend this simple technique. Right now, we will stop discussing the matter of applying convex functions to prove four-variable symmetric inequalities, and we will show a general theorem for the general case - problems in n variables. This useful theorem is often known as "Single inflection point Theorem" (or, for shortening, **SIP** theorem)

Theorem 5 (SIP theorem)**.** *Let f be a twice diffirentiable function on $\mathbb{R}$ with a single inflection point. For a fixed real number S, we denote*

$$g(x) = (n-1)f(x) + f\left(\frac{S-x}{n-1}\right).$$

For all real numbers $x_1, x_2, ..., x_n$ with sum S, we have

$$\inf_{x\in\mathbb{R}} g(x) \le f(x_1) + f(x_2) + ... + f(x_n) \le \sup_{x\in\mathbb{R}} g(x).$$

PROOF. First we will prove that

$$f(x_1) + f(x_2) + ... + f(x_n) \ge \inf_{x\in\mathbb{R}} g(x).$$

Assume that a is the single inflection point of $f(x)$. Denote $\mathbb{I}_1 = [a, +\infty)$ and $\mathbb{I}_2 = (-\infty, a]$. According to the hypothesis, we deduce that either $f(x)$ is convex on $\mathbb{I}_1$, concave on $\mathbb{I}_2$ or $f(x)$ is concave on $\mathbb{I}_1$, convex on $\mathbb{I}_2$. WLOG, assume that $f(x)$ is convex on $\mathbb{I}_1$ and concave on $\mathbb{I}_2$. If $x_1, x_2, ..., x_n \in \mathbb{I}_1$, we are done immediately by **Jensen** inequality. Otherwise, suppose that $x_1, x_2, ..., x_k \in \mathbb{I}_2$ and $x_{k+1}, x_{k+2}, ..., x_n \in \mathbb{I}_1$. Since $f(x)$ is concave on $\mathbb{I}_2$, by **Karamata** inequality (see one of the following articles), we conclude that

$$f(x_1) + f(x_2) + ... + f(x_k) \le (k-1)f(a) + f(x_1 + x_2 + ... + x_k - (k-1)a).$$

Since $f(x)$ is convex on $\mathbb{I}_1$, we conclude that

$$(k-1)f(a) + f(x_{k+1}) + f(x_{k+2}) + ... + f(x_n) \ge (n-1)f\left(\frac{ka + x_{k+1} + x_{k+2} + ... + x_n}{n-1}\right).$$

For $\alpha = x_1 + x_2 + ... + x_k - (k-1)a$ and $\beta = \dfrac{ka + x_{k+1} + x_{k+2} + ... + x_n}{n-1}$, we get

$$f(x_1) + f(x_2) + ... + f(x_n) \geq (n-1)f(\beta) = (n-1)f(\beta) + f\left(\frac{S-\beta}{n-1}\right).$$

This shows the desired result immediately. Similar proof for the remaining part.

Comment. According to this proof, we get the following result

★ *Let f be a twice diffirentiable function on $\mathbb{R}$ with a single inflection point (f convex on $\mathbb{I}_1$ and concave on $\mathbb{I}_2$). For all real numbers $x_1, x_2, ..., x_n$ with sum S, there exist numbers $\alpha \in \mathbb{I}_1$ and $\beta \in \mathbb{I}_2$ such that*

$$(n-1)f(\alpha) + f\left(\frac{S-\alpha}{n-1}\right) \leq f(x_1) + f(x_2) + ... + f(x_n) \leq (n-1)f(\beta) + f\left(\frac{S-\beta}{n-1}\right).$$

∇

With the help of this theorem, we can prove a lot of nice and hard inequalities.

Example 1.14.7. *Let a, b, c, d be positive real numbers such that $abcd = 1$. Prove that*

$$a^3 + b^3 + c^3 + d^3 + 12 \geq 2(a + b + c + d + abc + bcd + cda + dab).$$

(Pham Kim Hung)

SOLUTION. We have to prove that $f(x) + f(y) + f(z) + f(t) \geq 4$ where x, y, z, t are $\ln a, \ln b, \ln c, \ln d$ respectively and

$$f(x) = e^{3x} - 2e^x - 2e^{-x}.$$

Clearly

$$f''(x) = 9e^{3x} - 2e^x - 2e^{-x}.$$

Denote $t = e^x$. The equation $f''(x) = 0$ is equivalent to $9t^4 - 2t^2 - 2 = 0$, or $9 = \dfrac{2}{t^2} + \dfrac{2}{t^4}$. This has exactly one positive real root. That means $f(x)$ has a single inflection point. Therefore, according to **SIP** theorem, we may return to consider the initial problem in case $a = b = c$ and $d = -a^3$. In this case, the inequality becomes

$$3a^3 + a^{-9} + 12 \geq 2\left(3a + a^{-3} + a^3 + 3a^{-1}\right)$$

or

$$a^{12} - 6a^{10} + 12a^9 - 8a^8 - 2a^6 + 1 \geq 0.$$

This last inequality can be rewritten as

$$(a-1)^2(t^{10} + 2t^9 - 3t^8 + 4t^7 + 5t^6 + 6t^5 + 5t^4 + 4t^3 + 3t^2 + 2t + 1) \geq 0,$$

which is obvious. Therefore we are done and the equality holds for $a = b = c = d = 1$.

∇

Example 1.14.8. *Let a, b, c, d be positive real numbers with sum 4. Prove that*

$$9\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)+56 \geq 15\left(a^2+b^2+c^2+d^2\right).$$

(Pham Kim Hung)

SOLUTION. We have to prove that $f(a)+f(b)+f(c)+f(d) \geq 0$ where

$$f(x)=\frac{9}{x}-15x^2.$$

Since $f''(x)=\frac{18}{x^3}-30$ has exactly one positive real root, we infer that $f(x)$ has a single inflection point. Applying **SIP** theorem, we only need to consider the inequality in case $a=b=c=x \leq \sqrt[3]{\frac{18}{30}}$ and $d=4-3x$. In this case, the inequality becomes

$$g(x)=9\left(\frac{3}{x}+\frac{1}{4-3x}\right)-15\left(3x^2+(4-3x)^2\right)+56.$$

Clearly

$$\begin{aligned} g'(x) &= 27\left(\frac{1}{(4-3x)^2}-\frac{1}{x^2}\right)-90\left(x-(4-3x)\right) \\ &= \frac{9(4-4x)}{x^2(4-3x)^2}\left(3(4-2x)-10x^2(4-3x)^2\right) \\ &= \frac{72(x-1)(3x-1)(6+15x-35x^2+15x^3)}{x^2(4-3x)^2}. \end{aligned}$$

Since $x \leq \sqrt[3]{\frac{18}{30}}<1$, we get that $6+15x-35x^2+15x^3>0$. Therefore, in the range $(0\ ,\ 1]$, the minimum of g is attained at $x=\frac{1}{3}$. In other words, we can conclude that

$$g(x) \geq g\left(\frac{1}{3}\right)=0.$$

This ends the proof. Equality holds for $a=b=c=\frac{1}{3}, d=3$ or permutations.

∇

Example 1.14.9. *Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1a_2...a_n=1$. Prove that*

$$\frac{1}{n-1+a_1}+\frac{1}{n-1+a_2}+...+\frac{1}{n-1+a_n} \leq 1.$$

(Romania TST 1998)

SOLUTION. We have to prove that

$$f(x_1)+f(x_2)+...+f(x_n)\le 1$$

where $x_i=\ln a_i\ \forall i\in\{1,2,...,n\}$ and

$$f(x)=\frac{1}{n-1+e^x}.$$

We have

$$f''(x)=\frac{2\left(e^x-(n-1)\right)}{(n-1+e^x)^3}.$$

Since the function $f''(x)$ has exactly one root, according to **SIP** theorem, we conclude that it suffices to consider the inequality in case $x_1=x_2=...=x_{n-1}$. In other words, we have to prove that if $a_1=a_2=...=a_{n-1}=a$ and $a_n=a^{1-n}$ then

$$\frac{n-1}{n-1+a}+\frac{1}{a^{1-n}+n-1}\le 1.$$

This can be reduced to

$$(a-1)^2\left(n(n-1)\sum_{i=0}^{n-2}a^i-n\sum_{i=0}^{n-2}a^i\right)\ge 0,$$

which is obvious. Equality holds for $a_1=a_2=...=a_n=1$.

∇

Example 1.14.10. *Let $a_1,a_2,...,a_n$ be positive real numbers with product 1. Prove that*

$$a_1^2+a_2^2+...+a_n^2-n\ge\frac{2n}{n-1}\sqrt[n]{n-1}(a_1+a_2+...+a_n-n).$$

(Gabriel Dospinescu, Calin Popa)

SOLUTION. For $k=\dfrac{2n}{n-1}\sqrt[n]{n-1}$, we consider the following function

$$f(x)=e^{2x}-ke^x.$$

We have to prove that $f(x_1)+f(x_2)+...+f(x_n)\ge(1-k)n$ where $x_i=\ln a_i\ \forall i\in\{1,2,...,n\}$. Since the function

$$f''(x)=4e^{2x}-ke^x$$

has exactly one real root, we infer that $f(x)$ has a single inflection point. According to the **SIP** theorem, we may assume that $x_1=x_2=...=x_{n-1}$, or in other words, $a_1=a_2=...=a_{n-1}$. The rest follows from what we have done in example **??**. The equality holds for $a_1=a_2=...=a_n=1$.

∇

Example 1.14.11. *Let a, b, c, d, e, f be positive real numbers with sum 6. Prove that*

$$(1+a^2)(1+b^2)(1+c^2)(1+d^2)(1+e^2)(1+f^2) \geq (1+a)(1+b)(1+c)(1+d)(1+e)(1+f).$$

(Pham Kim Hung)

SOLUTION. Consider the following function in the positive variable x

$$f(x) = \ln(1 + x^2) - \ln(1 + x).$$

We have certainly

$$f''(x) = \frac{2(1 - x^2)}{(1 + x^2)^2} + \frac{1}{(1 + x)^2}.$$

The equation $f''(x) = 0$ is equivalent to

$$g(x) = 3x^4 + 4x^3 + 2x^2 - 4x - 1 = 0.$$

Since $g''(x) > 0\ \forall x > 0$, the equation $g(x) = 0$ has no more than two positive real roots. However, if it had exactly two positive real roots, it must have one more root (because the last coefficient is -1). So we get that $g(x)$ has exactly one positive real root. In other words, $f(x)$ has a single inflection point. According to **SIP** theorem, we only need to consider the initial inequality in case $a = b = c = d = e = x$ and $e = 6 - 5x$. We have to prove that

$$p(x) = 5\ln(1 + x^2) + \ln\left(1 + (6 - 5x)^2\right) - 5\ln(1 + x) + \ln(7 - 5x) \geq 0.$$

Clearly,

$$\begin{aligned} p'(x) &= \frac{10x}{1 + x^2} - \frac{10(6 - 5x)}{1 + (6 - 5x)^2} - \frac{5}{1 + x} + \frac{5}{7 - 5x} \\ &= 30(x - 1)\left(\frac{2 - 2x(6 - 5x)}{(1 + x^2)(1 + (6 - 5x)^2)} + \frac{1}{(1 + x)(7 - 5x)}\right). \end{aligned}$$

Consider the function

$$\begin{aligned} q(x) &= (1 + x^2)(1 + (6 - 5x)^2) + (2 - 2x(6 - 5x))(1 + x)(7 - 5x) \\ &= -25x^4 + 20x^3 + 98x^2 - 140x + 51. \end{aligned}$$

We will prove that $q(x) \geq 0\ \forall 0 \leq x \leq \frac{6}{5}$. Indeed, if $x \geq 1$ then $x(6 - 5x) \leq 1 \Rightarrow q(x) \geq 0$. Otherwise, if $x \leq 1$, consider the following cases

Case $x \leq 0.8$. We are done since

$$q(x) = 10x(4 - 5x) + (98x^2 - 140x + 51) > 0.$$

Case $0.8 \leq x \leq 0.88$. We are done since

$$q(x) = x^2(2 + 20x - 25x^2) + (96x^2 - 140x + 51) \geq 0.$$

Case $0.88 \le x \le 1$. We are done since

$$q(x) = x^2(5 + 20x - 25x^2) + (94x^2 - 140x + 51) \ge 0.$$

In every case, it is clear that $q(x) \ge 0$. We conclude that $p'(x) = 0 \Leftrightarrow x = 1$, which implies $p(x) \ge p(1) = 0$. This ends the proof, and the equality holds for $a = b = c = d = e = f = 1$.

∇

Example 1.14.12. *Let a, b, c, d be positive real numbers with sum 4. Prove that*

$$(1 + a^2)(1 + b^2)(1 + c^2)(1 + d^2) \ge \frac{10^4}{9^3}.$$

(Pham Kim Hung, Volume I)

SOLUTION. We have to prove that

$$f(a) + f(b) + f(c) + f(d) \ge 4\ln 10 - 3\ln 9,$$

where $f(x) = \ln\left(1 + x^2\right)$. Since

$$f''(x) = \frac{2(1 - x^2)}{(1 + x^2)^2}$$

has exactly one positive real root $x = 1$, we obtain by **SIP** theorem that there exists a numbers $p \le 1$ for which

$$f(a) + f(b) + f(c) + f(d) \ge 3f(p) + f(4 - 3p).$$

Denote

$$g(p) = 3f(p) + f(4 - 3p) = 3\ln\left(1 + p^2\right) + \ln\left(1 + (4 - 3p)^2\right),$$

then we get

$$g'(p) = \frac{6p}{1 + p^2} - \frac{6(4 - 3p)}{1 + (4 - 3p)^2} = \frac{24(p - 1)^2(3p - 1)}{(1 + p^2)\left(1 + (4 - 3p)^2\right)},$$

and it is easy to conclude that

$$g(p) \ge g\left(\frac{1}{3}\right) = 4\ln 10 - 3\ln 9,$$

as desired. The equality holds for $a = b = c = \frac{1}{3}, d = 3$ or permutations.

∇

1.15 nSMV Theorem and Applications

If the importance of **SMV** theorem is to provide a standard way to prove four-variable inequalities, the improved **SMV** theorem, called n**SMV**, becomes very effective in proving n-variables inequalities. Although n**SMV** theorem is based on **SMV** theorem, the intensive applications of n**SMV**are really incredible. To prove n**SMV** theorem, we use a new result similar to the *general mixing variable lemma* (the result shown in the previous article), but first, we need to clarify some basic properties that hold between three real numbers.

Lemma 1. *Suppose that a, b, c are non-negative real numbers satisfying $a + b + c = 2 + r$ and $a^2 + b^2 + c^2 = 2 + r^2,\ r \le 1$ is non-negative real constant then $abc \ge r$.*

Lemma 2. *Suppose that m, n are two non-negative real constants, then the system of equations*

$$\begin{cases} x + y + z = m \\ xy + yz + zx = n \end{cases}$$

has a solution $(x, y, z) = (a, b, c)$, with $a, b, c \ge 0$ if and only if $m^2 \ge 3n$.

PROOF. The second lemma is quite obvious, therefore we will prove the first one.

Denote $x = a-1, y = b-1, z = c-1$ then $x+y+z = r-1$ and $x^2+y^2+z^2 = (r-1)^2$, so we infer $xy+yz+zx = 0$. We also have that x, y, z are the real roots of the polynomial $f(t) = (t - x)(t - y)(t - z) = t^3 + (1 - r)t^2 - xyz$. Now suppose that $xyz < 0$ then all coefficients of $f(t)$ are non-negative and therefore all its roots are non-positive, or $x, y, z \le 0$. This result contradicts the assumption $xy + yz + zx = 0$. Then we must have $xyz = (a - 1)(b - 1)(c - 1) \ge 0$ and the conclusion follows.

∇

Lemma 3. *Suppose that a, b, c are three non-negative real numbers satisfying $a + b + c = m$, $ab + bc + ca = n$, where m and n are two non-negative real constants. If $m^2 \ge 4n$ then the minimum value of abc is 0, with equality for $a = 0$ and $(b, c) = \left(\dfrac{m + \sqrt{m^2 - 4n}}{2}, \dfrac{m - \sqrt{m^2 - 4n}}{2}\right)$ up to permutation. If $3n \le m^2 < 4n$ then the minimum values of abc are attained for $(a, b, c) = \left(\dfrac{m - 2\sqrt{m^2 - 3n}}{3}, \dfrac{m + \sqrt{m^2 - 3n}}{3}, \dfrac{m + \sqrt{m^2 - 3n}}{3}\right)$ up to permutation.*

PROOF. We consider the following cases

(i) The first case. If $m^2 \le 4n$ then there exist two numbers a_0, b_0 such that $a + b =$

$m, ab = 4n$, therefore $(a, b, c) = (a_0, b_0, 0)$ satisfies the system

$$\begin{cases} a+b+c=m \\ ab+bc+ca=n \end{cases}$$

Certainly, the $a_0 \cdot b_0 \cdot 0 = 0$ and therefore the minimum of abc is 0.

(ii) The second case. If $3n \le m^2 < 4n$ then it is easy to check that there exist two numbers k, r $(k \ge r \ge 0)$ for which $m = 2k + r, n = 2k^2 + r^2$. According to lemma 1, we conclude that $abc \ge k^2 r$, as desired.

∇

As a matter of fact, the third lemma can be proved easily by derivatives, but proofs without derivatives are better. However, derivatives can help us confirm easily that

Lemma 4. *Suppose that a, b, c are three non-negative real numbers satisfying $a + b + c = m$, $ab + bc + ca = n$, where m and n are two non-negative real constants and $m^2 \ge 3n$. The maximum value of abc is attained for $(a, b, c) = \left(\frac{m+2\sqrt{m^2-3n}}{3}, \frac{m-\sqrt{m^2-3n}}{3}, \frac{m-\sqrt{m^2-3n}}{3}\right)$ up to permutation.*

Lemma 5. *Suppose that a, b, c are three positive real numbers satisfying $a + b + c = m$, $1/a + 1/b + 1/c = n$, where m and n are two positive real constants and $mn \ge 9$, then abc is maximal when $a = b = \frac{(mn+3)+\sqrt{(mn-1)(mn-9)}}{4n}$, $c = \frac{(mn-3)-\sqrt{(mn-1)(mn-9)}}{2n}$ up to permutation; and abc is minimal when $a = b = \frac{(mn+3)-\sqrt{(mn-1)(mn-9)}}{4n}$, $c = \frac{(mn-3)+\sqrt{(mn-1)(mn-9)}}{2n}$ up to permutation.*

Although these lemma seem to be hard to apply, they are meant to be used for a more important result, n**SMV** theorem.

Before showing n**SMV** theorem, we will prove an improved *general mixing variable lemma* (and a give general kind of Δ transformation). Its proof is still based on that of the initial lemma.

Lemma 6 (Improved general mixing variable lemma)**.** *Let $(a_1, a_2, ..., a_n)$ be a sequence of real numbers and ϵ_1, ϵ_2 be two constants such that $\epsilon_1, \epsilon_2 \in (0, 1)$. Carry out the following transformations*

1. Choose $i, j \in \{1, 2, ..., n\}$ to be two different indices satisfying

$$a_i = \max(a_1, a_2, ..., a_n), a_j = \min(a_1, a_2, ..., a_n).$$

2. Replace a_i, a_j by a certain number α (without changing their ranks) for which

$$\alpha \in [a_i, a_j], \left|\frac{a_i - \alpha}{a_i - a_j}\right| < \epsilon_1 < 1, \left|\frac{a_i - \alpha}{a_i - a_j}\right| < \epsilon_2 < 1.$$

After repeating these two transformations, all numbers a_i tend to the same limit.

PROOF. The following proof is based on the proof of the general mixing variables lemma. Henceforward, we will call this transformation the Γ-transformation, which is indeed an extension of the initial Δ transformation. Denote the first sequence as $(a_{1,1}, a_{1,2}, ..., a_{1,n})$ and the sequence after the k-th Γ-transformation as $(a_{k,1}, a_{k,2}, ..., a_{k,n})$. Denote $M_k = \max(a_{k,1}, a_{k,2}, ..., a_{k,n})$ and $m_k = \min(a_{k,1}, a_{k,2}, ..., a_{k,n})$. We have of course that $(M_k)_{k=1}^{\infty}$ is a non-increasing sequence and $(m_k)_{k=1}^{\infty}$ is a non-decreasing sequence, so there are two finite limits

$$M = \lim_{k\to\infty} M_k,\ m = \lim_{k\to\infty} m_k$$

and we need to prove that $M = m$. WLOG, suppose that $M_1 = a_1 = a_{1,1}$ and $m_1 = a_n = a_{n,1}$. The Γ-transformation changes a_1 and a_n to $a_{2,1} = a_{2,n} = x_2 \in [a_1, a_n]$ (we can assume that $a_1 > a_n$). If there exists some transformations that transform $a_{n,1}$ again (that means there exists some numbers $k > 2$ for which m_k (or M_k) is equal to x_2), we must have m_2 (or M_2) is equal to x_2. Indeed, suppose that $m_k = x_2$, since $x_2 = a_{2,n}$ it follows that $x_2 \geq m_2 \geq ... \geq m_k$ and the equality must hold, or $m_2 = x_2$. Denote x_k to be the result of M_k and m_k after the k-th Γ-transformation, then we infer that

$$S = \{k : \exists l > k | m_l = x_k\} = \{k | m_{k+1} = x_k\}\ ;$$

$$P = \{k : \exists l > k | M_l = x_k\} = \{k | M_{k+1} = x_k\}\ .$$

By hypothesis, if $k \in S$ then

$$M_{k+1} - m_{k+1} = M_{k+1} - x_k \leq M_k - x_k \leq \epsilon_1(M_k - m_k).$$

Similarly, if $k \in P$ then

$$M_{k+1} - m_{k+1} \leq \epsilon_2(M_k - m_k).$$

Because $\epsilon_1, \epsilon_2 \in (0, 1)$, if S or P are infinite, we have $\lim\limits_{k\to\infty}(M_k - m_k) = 0 \Rightarrow M = m$ and the conclusion follows. Otherwise, both S and P are finite. We deduce that Γ-transformations don't impact on $a_{k,1}$ and $a_{k,n}$ after a sufficiently large number k. Without loss of generality, we can assume that $S = P = \emptyset$. Therefore $a_{k,1} = a_{2,1}$ for all number $k \in \mathbb{N}$ and $k \geq 2$ and we can eliminate the number $a_{2,1}$ from the sequence and consider the remaining problem for the sequence $(a_{2,2}, a_{2,3}, ..., a_{2,n})$ (only $n - 1$ terms). By a simple induction, we have the desired result.

∇

From this lemma, we can deduce a generalization of **SMV** theorem as follows

Theorem 6 (n**SMV** theorem)**.** *Suppose that the function $f : \mathbb{R}^n \to \mathbb{R}$ is a continuous, symmetric, lower bounded function satisfying the condition*

$$f(a_1, a_2, ..., a_n) \geq f(b_1, b_2, ..., b_n),$$

where $(b_1, b_2, ..., b_n)$ is obtained from $(a_1, a_2, ..., a_n)$ by a Γ-transformation, then

$$f(a_1, a_2, ..., a_n) \geq f(x, x, ..., x)$$

where x is a certain number (normally defined by the specific form of Γ).

The basic importance of n**SMV** theorem is that it uses a very general transformation Γ that has a lot of particular applications. Indeed, here are some of them

Corollary 1. *Suppose that $x_1, x_2, ..., x_n$ are positive real numbers such that*

$$x_1 + x_2 + ... + x_n = const, \frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} = const$$

and $f(x_1, x_2, ..., x_n)$ a continuous, symmetric, lower bounded function satisfying that, if $x_1 \geq x_2 \geq ... \geq x_n$ and $x_2, x_3, ..., x_{n-2}$ are fixed then $f(x_1, x_2, ..., x_n) = g(x_1, x_{n-1}, x_n)$ is a strictly increasing function of $x_1x_{n-1}x_n$; then $f(x_1, x_2, ..., x_n)$ attains the minimum value if and only if $x_1 = x_2 = ... = x_{n-1} \leq x_n$. If $x_1 \geq x_2 \geq ... \geq x_n$ and $x_3, ..., x_{n-1}$ are fixed then $f(x_1, x_2, ..., x_n) = g(x_1, x_2, x_n)$ is a strictly increasing function of $x_1x_2x_n$; then $f(x_1, x_2, ..., x_n)$ attains the maximum value if and only if $x_1 = x_2 = ... = x_{n-1} \geq x_n$.

Corollary 2. *Suppose that $x_1, x_2, ..., x_n$ are non-negative real numbers satisfying*

$$x_1 + x_2 + ... + x_n = const, x_1^2 + x_2^2 + ... + x_n^2 = const$$

and $f(x_1, x_2, ..., x_n)$ a continuous, symmetric, under-limitary function satisfying that, if $x_1 \geq x_2 \geq ... \geq x_n$ and $x_2, x_3, ..., x_{n-2}$ are fixed then $f(x_1, x_2, ..., x_n) = g(x_1, x_{n-1}, x_n)$ is a strictly increasing function of $x_1x_{n-1}x_n$; then $f(x_1, x_2, ..., x_n)$ attains the minimum value if and only if $x_1 = x_2 = ... = x_k = 0 < x_{k+1} \leq x_{k+2} = ... = x_n$, where k is a certain natural number and $k < n$. If $x_1 \geq x_2 \geq ... \geq x_n$ and $x_3, ..., x_{n-1}$ are fixed then $f(x_1, x_2, ..., x_n) = g(x_1, x_2, x_n)$ is a strictly increasing function of $x_1x_2x_n$; then $f(x_1, x_2, ..., x_n)$ attains the maximum value if and only if $x_1 = x_2 = ... = x_{n-1} \leq x_n$.

Proof. To prove the above corollaries, we only show the hardest, that is the second part of the second corollary (and other parts are proved similarly).

★ *Suppose that $x_1, x_2, ..., x_n$ are non-negative real numbers satisfying*

$$x_1 + x_2 + ... + x_n = const, x_1^2 + x_2^2 + ... + x_n^2 = const$$

and $f(x_1, x_2, ..., x_n)$ a continuous, symmetric, under-limitary function satisfying that if $x_1 \geq x_2 \geq ... \geq x_n$ and $x_2, x_3, ..., x_{n-2}$ are fixed then $f(x_1, x_2, ..., x_n) = g(x_1, x_{n-1}, x_n)$

is a strictly increasing function of $x_1x_{n-1}x_n$*; then* $f(x_1, x_2, ..., x_n)$ *attains the minimum value if and only if* $x_1 = x_2 = ... = x_k = 0 < x_{k+1} \le x_{k+2} = ... = x_n$*, where* $k < n$ *is a certain natural number.*

To prove this one, we will chose the transformation Γ on $(x_1, x_2, ..., x_n)$ as

(i). The first step. Choose $i, j, k \in \{1, 2, ..., n\}$ to be different indices satisfying

$$a_i = \max(a_1, a_2, ..., a_n),\ a_j = \min_{t=\overline{1,n}, a_t>0}\{a_t\},\ a_k = \min_{t=\overline{1,n}, a_t>0, t\ne j}\{a_t\}.$$

(ii). The second step. With $s = a+b+c, p = ab+bc+ca$, replace a_i, a_j, a_k by

+, $a'_i = a'_k = \dfrac{s+\sqrt{s^2-3p}}{3}, a'_j = \dfrac{s-2\sqrt{s^2-3p}}{3}$ if $4p \ge s^2 \ge 3p$ (1).

+, $a'_i = \dfrac{s+\sqrt{s^2-4p}}{2}, a'_k = \dfrac{s-\sqrt{s^2-4p}}{2}, a'_j = 0$ if $s^2 > 4p$ (2).

After each of these transformations, (a_i, a_j, a_k) becomes a new triple (a'_i, a'_j, a'_k) with

$$a_i + a_j + a_k = a'_i + a'_j + a'_k,\ a_ia_j + a_ja_k + a_ka_i = a'_ia'_j + a'_ja'_k + a'_ka'_i,$$

but the product $a'_ia'_ja'_k$ is minimal (that is $a_ia_ja_k \ge a'_ia'_ja'_k$). Notice that the step (2) can't be carried indefinitely, because we can change positive terms of $(a_1, a_2, ..., a_n)$ to 0 in only finitely many times (no more than n times). Therefore, to get the conclusion, we only need to prove that if $4p \ge s^2 \ge 3p$ then

$$\left|\frac{a'_i - a_k}{a_i - a_k}\right| < \epsilon_1 < 1,\ \left|\frac{a'_i - a_i}{a_i - a_k}\right| < \epsilon_2 < 1.$$

We don't need to take care of how a_j changes, because the condition $4p \ge s^2 \ge 3p$ ensures that $a'_j \ge 0$. Denote $a_i = a, a_k = b, a_j = c$ so $a \ge b \ge c$ and therefore

$$a'_i = \frac{a+b+c+\sqrt{a^2+b^2+c^2-ab-bc-ca}}{3}$$

and we deduce that

$$\begin{aligned}\left|\frac{a'_i - a_k}{a_i - a_k}\right| &= \frac{a+b+c+\sqrt{a^2+b^2+c^2-ab-bc-ca}-3b}{3(a-b)}\\ &= \frac{(a-b)-(b-c)+\sqrt{\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)}}{3(a-b)}\\ &\le \frac{(a-b)-(b-c)+(a-c)}{3(a-b)} = \frac{2}{3} < 1.\end{aligned}$$

$$\begin{aligned}\left|\frac{a'_i - a_i}{a_i - a_k}\right| &= \frac{(a-b)+(a-c)-\sqrt{\frac{1}{2}\left((a-b)^2+(b-c)^2+(c-a)^2\right)}}{3(a-b)}\\ &\le \frac{(a-b)+(a-c)-(b-c)}{3(a-b)} = \frac{2}{3} < 1.\end{aligned}$$

Notice that the Γ-transformation makes the product $a_i a_j a_k$ minimal, and also $f(a_1, a_2, ..., a_n)$ minimal (if we fix all numbers $a_t, t \neq i, j, k$). Therefore we are done.

∇

According to this proof, we can prove the following result as well (that let's us use n**SMV** theorem more freely) as follows

Corollary 3. *Suppose that $x_1, x_2, ..., x_n$ are non-negative real numbers such that*

$$x_1 + x_2 + ... + x_n = const, x_1^2 + x_2^2 + ... + x_n^2 = const.$$

Let $f(x_1, x_2, ..., x_n)$ be a continuous, symmetric, under-limitary function. If we fix $x_4, x_5, ..., x_n$ then $f(x_1, x_2, ..., x_n) = g(x_1, x_2, x_3)$ is a strictly increasing function of $x_1x_2x_3$ then $f(x_1, x_2, ..., x_n)$ attains the minimum value if and only if $x_1 = x_2 = ... = x_k = 0 < x_{k+1} \leq x_{k+2} = ... = x_n$, where $k < n$ is a certain natural number. If we fix $x_4, x_5, ..., x_n$ then $f(x_1, x_2, ..., x_n) = g(x_1, x_2, x_3)$ is a strictly increasing function of $x_1x_2x_3$ then $f(x_1, x_2, ..., x_n)$ attains the maximum value if and only if $x_1 = x_2 = ... = x_{n-1} \geq x_n$.

Actually, I know that these results are difficult for you to comprehend because of their complicated appearances, but you will see that everything is clear after you try to prove the following examples

Example 1.15.1. *Let a, b, c, d be non-negative real numbers. Prove that*

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2.$$

(Turkevici's inequality)

SOLUTION. If we fix $a + b + c = const$ and $a^2 + b^2 + c^2 = const$, then

$$\text{RHS} - \text{LHS} = (a^2+b^2+c^2)^2 - 3(ab+bc+ca)^2 + 6abc(a+b+c) + 2abcd - d^2(a^2+b^2+c^2)$$

is certainly an increasing function of abc. By corollary 3, it's enough to prove the inequality if $a = b = c \geq d$ or $abcd = 0$. If $d = 0$ then we have to prove that $a^4 + b^4 + c^4 \geq a^2b^2 + b^2c^2 + c^2a^2$, which is obvious. If $a = b = c$, the inequality becomes $3d^4 + 2a^3d \geq 3a^2d^2$, which is directly obtained from **AM-GM** inequality, too. The proof is completed successfully.

∇

Example 1.15.2. *Let a, b, c, d be non-negative real numbers with sum 4. Prove that*

$$(1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) \leq 10(a^2 + b^2 + c^2 + d^2) + 41abcd.$$

(Pham Kim Hung)

SOLUTION. First, notice that if $a \le b \le c \le d$ and $c \le 1/3$ then $a, b \le 1/3$ and $d \ge 3$ and we are done because

$$10(a^2 + b^2 + c^2 + d^2) + 41abcd \ge 90 > (1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d).$$

So we may assume that $a \le b \le c \le d, c \ge \frac{1}{3}$. We fix $c = const$, $a^2+b^2+c^2+d^2 = const$ (therefore $a + b + d, a^2 + b^2 + d^2$ are fixed, too). Denote

$$f(a, b, c, d) = (1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) - 10(a^2 + b^2 + c^2 + d^2) - 41abcd.$$

The coefficient of abd is $8(1 + 2c) - 41c = 8 - 25c \le 0$, so f is a strictly decreasing function of abc. According to corollary 2 of n**SMV** theorem, it's enough to consider the two cases $a \le b = c = d$ and $abcd = 0$. If $a \le b = c = d = x$ then the inequality can be rewritten as $(x - 1)^2(-75x^2 + 14x + 151) \ge 0$, which is true because $x \le 4/3$. If $abcd = 0$, then $a = 0$ and the inequality is also obvious because

$$10(b^2 + c^2 + d^2) \ge \frac{160}{3} \ge (1 + 2a)(1 + 2b)(1 + 2c).$$

We are done and the equality holds for $a = b = c = d = 1$.

Comment. In the same manner, we can prove a stronger result as follows

★ *Let a, b, c, d be non-negative real numbers with sum 4 then*

$$144(1 + 2a)(1 + 2b)(1 + 2c)(1 + 2d) \le 1331(a^2 + b^2 + c^2 + d^2) + 6340abcd.$$

∇

Example 1.15.3. *Let $x_1, x_2, ..., x_n$ be non-negative real numbers such that $x_1 + x_2 + ... + x_n = n$. Prove that*

$$(x_1x_2...x_n)^{\frac{1}{\sqrt{n-1}}}(x_1^2 + x_2^2 + ... + x_n^2) \le n.$$

(Vasile Cirtoaje)

SOLUTION. If we fix $x_1+x_2+x_3$ and $x_1^2+x_2^2+x_3^2$ then the left-hand expression of the above inequality is clearly a strictly increasing function of $x_1x_2x_3$, so, according to the corollary 2 of n**SMV** theorem, we conclude that it suffices to consider the initial inequality in case $x_1 = x_2 = ... = x_{n-1} = x \le 1$ and $x_n = n - (n - 1)x$. In this case, the inequality becomes

$$x^{\sqrt{n-1}}(n - (n - 1)x)^{\frac{1}{\sqrt{n-1}}}\left((n - 1)x^2 + (n - (n - 1)x)^2\right) \le n$$

or $f(x) \le 0\ \forall x \le 1$ where

$$f(x) = \sqrt{n-1}\ln x + \frac{1}{\sqrt{n-1}}\ln(n-(n-1)x) + \ln\left((n - 1)x^2 + (n - (n - 1)x)^2\right) - \ln n.$$

It is easy to check that

$$f'(x) = \frac{n\sqrt{n-1}(1-x)\left(\sqrt{n-1}x - n + (n-1)x\right)^2}{x(n-(n-1)x)\left(nx^2 + (n-(n-1)x)^2\right)}.$$

Since $x \le 1$, we get that $f'(x) \ge 0$, therefore

$$f(x) \le f(1) = 0.$$

This ends the proof. The equality holds for $x_1 = x_2 = ... = x_n = 1$.

$$\nabla$$

Example 1.15.4. *Let a, b, c, d be positive real numbers such that that*

$$2(a+b+c+d)^2 = 5(a^2+b^2+c^2+d^2).$$

Find the minimum value of

$$P = \frac{a^4+b^4+c^4+d^4}{abcd}.$$

(Pham Kim Hung)

SOLUTION. First we guess that the equality holds for $b = c = d$ (in this case, we find out $a = 2+\sqrt{5}, b = c = d = 1$ and permutations) and this is the key to the solution. Indeed, denote $k = (2+\sqrt{5})^3 + (2+\sqrt{5})^{-1} > 78$, we will prove

$$a^4 + b^4 + c^4 \ge 4abcd.$$

WLOG, assume that $a \ge b \ge d \ge c$. We fix d and suppose that $a+b+c = const, a^2 + b^2 + c^2 = const$. By hypothesis, we have

$$4(a+b+c+d)^2 = 10(a^2+b^2+c^2+d^2) \ge 5\left((a+b)^2 + (c+d)^2\right)$$

so

$$(a+b)^2 + (c+d)^2 \le 8(a+b)(c+d) \;\Rightarrow a+b \le (4+\sqrt{15})(c+d)$$

and therefore

$$a+b < \left(4+\sqrt{15}\right)(c+d) < \left(8+2\sqrt{15}\right)d < 18d \;\Rightarrow 4(a+b+c) \le 76d.$$

Moreover, notice that

$$a^4+b^4+c^4 - kd \cdot abc = (a^2+b^2+c^2)^2 - 2(ab+bc+ca)^2 + [4(a+b+c) - kd]\, abc$$

is a stricly decreasing function of abc (because $a+b+c$ and $a^2+b^2+c^2$ have been fixed already). By the second corollary, it suffices to consider the initial problem in case $a \ge b = c = d$. If $b = c = d$, we deduce that $a = \left(2+\sqrt{5}\right)b$ by hypothesis, so

$$a^4+b^4+c^4+d^4 = \left((2+\sqrt{5})^4 + 1\right)b^4 = kabcd.$$

$$\nabla$$

Example 1.15.5. *Suppose that $a_1, a_2, ..., a_n$ are positive real numbers satisfying*

$$a_1 + a_2 + ... + a_n = {a_1}^{-1} + {a_2}^{-1} + ... + {a_n}^{-1} = n + 2.$$

Find the minimum and maximum value of

$$P = a_1^2 + a_2^2 + ... + a_n^2 + 2a_1a_2...a_n.$$

(Pham Kim Hung)

SOLUTION. Without loss of generality, we may assume that $a_1 \geq a_2 \geq ... \geq a_{n-1} \geq a_n$. We fix $s = a_1 + a_2 + a_n$ and $r = \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_n}$. Denote $x = a_1a_2a_n$ and $p = a_3a_4...a_{n-1}$ then the expression P can be rewritten into the form

$$P = s^2 - 2rx + 2px + \sum_{i=3}^{n-1} a_i^2 = s^2 - (2r - 2p)x + \sum_{i=3}^{n-2} a_i^2.$$

Now we will prove that $r \geq p$, or

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_n} \geq a_3a_4...a_{n-2}.$$

Indeed, according to the assumptions, **AM-GM** inequality indicates that

$$n + 2 = a_1 + a_2 + ... + a_n \geq n\sqrt[n]{a_1a_2...a_n}$$

$$\Rightarrow a_1a_2...a_n \leq \left(1 + \frac{2}{n}\right)^n \leq e^2 < 9$$

$$\Rightarrow a_3...a_{n-1} \leq \frac{9}{a_1 + a_2 + a_n} \leq a_1 + a_2 + a_n.$$

This result shows that $P = P(x)$ is a strictly decreasing function of x. By corollary 1 of n**SMV** theorem, we deduce that P attains the maximum if $x_1 \geq x_2 = x_3 = ... = x_n$ and attains the minimum value if $x_1 \leq x_2 = x_3 = ... = x_n$. By hypothesis, it's easy to determine the values of $(x_i)_{i=1}^n$ for each of these cases. The proof is completed.

∇

Example 1.15.6. *Let a, b, c, d be non-negative real numbers with sum 1. Prove that*

$$2(a^2 + b^2 + c^2 + d^2) \geq 27\sqrt[3]{(a^2 + b^2)(a^2 + c^2)(a^2 + d^2)(b^2 + c^2)(b^2 + d^2)(c^2 + d^2)}.$$

(Pham Kim Hung)

SOLUTION. We will prove a homogeneous inequality as follows

$$2\left(\sum_{cyc} a\right)^2 \left(\sum_{cyc} a^2\right) \geq 27\sqrt[3]{\prod_{\text{sym}}(a^2 + b^2)}.$$

WLOG, assume that $a \geq b \geq c \geq d$. If we fix $a+c+d, a^2+c^2+d^2$ and let $acd = x$ then

$$(a^2+c^2)(c^2+d^2)(d^2+a^2) = (a^2+c^2+d^2)(ac+cd+da)^2 - 2(a^2+c^2+d^2)(a+c+d)x - x^2$$

is a stricly decreasing function of x. Moreover,

$$(b^2+a^2)(b^2+c^2)(b^2+d^2) = b^6+b^4(a^2+c^2+d^2)+b^2(ac+cd+da)^2 - 2b^2(a+c+d)x + x^2$$

is also a strictly decreasing function of x because its derivative is

$$-2b^2(a+d+d) + 2x \geq 2b^2a - 2x \leq 0.$$

The expression $\prod_{cyc}(a^2+b^2)$ is a strictly decreasing function of $x = acd$. By corollary 2 of n**SMV** theorem, we conclude that it suffices to examine the initial problem in cases $a = b = c$ or $d = 0$. If $d = 0$, the inequality becomes

$$8(a+b+c)^6(a^2+b^2+c^2)^3 \geq 3^9a^2b^2c^2(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

This one follows from **AM-GM** inequality immediately because

$$(a+b+c)^6 \geq 3^6a^2b^2c^2$$

$$8(a^2+b^2+c^2)^3 \geq 27(a^2+b^2)(b^2+c^2)(c^2+a^2).$$

Consider the case $a = b = c$. The inequality becomes

$$(3a+d)^2(3a^2+d^2) \geq 27a^2(a^2+d^2)$$

which is obvious because

$$(3a+d)^2(3a^2+d^2) \geq (9a^2+6d^2)(3a^2+d^2) \geq 27a^2(a^2+d^2).$$

This is the end of the proof. The equality holds for $a = b = c, d = 0$ or permutations.

Comment. Using this result, we get the following one

★ *If a, b, c, d are non-negative real numbers then*

$$\frac{1}{a^2+b^2} + \frac{1}{a^2+c^2} + \frac{1}{a^2+d^2} + \frac{1}{b^2+c^2} + \frac{1}{b^2+d^2} \geq \frac{81}{2(a+b+c+d)^2}.$$

Indeed, just apply **AM-GM** inequality, and we can conclude that

$$\sum_{cyc} \frac{1}{a^2+b^2} = \sum_{cyc} \left(\frac{1}{a^2+b^2} + \frac{1}{c^2+d^2} \right)$$

$$= \left(\sum_{cyc} a^2 \right) \left(\sum_{cyc} \frac{1}{(a^2+b^2)(c^2+d^2)} \right)$$

$$\geq \frac{3(a^2+b^2+c^2+d^2)}{\sqrt[3]{\prod_{\text{sym}}(a^2+b^2)}} \geq \frac{81}{2(a+b+c+d)^2}.$$

∇

Example 1.15.7. *Let $x_1, x_2, ..., x_n$ be positive real numbers such that $x_1+x_2+...+x_n = n$. Prove that*

$$\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}+\frac{2n\sqrt{n-1}}{x_1^2+x_2^2+...+x_n^2} \geq n+2\sqrt{n-1}.$$

(Pham Kim Hung)

SOLUTION. We fix the sums $x_1+x_2+x_3, x_1^2+x_2^2+x_3^2$ and fix the $n-3$ numbers $x_3, x_4, ..., x_n$. Clearly, $x_1x_2+x_2x_3+x_3x_1$ is a constant and

$$\frac{1}{x_1}+\frac{1}{x_2}+\frac{1}{x_3} = \frac{x_1x_2+x_2x_3+x_3x_1}{x_1x_2x_2}$$

is a decreasing function of $x_1x_2x_3$. By corollary 2, we only need to consider the inequality in case $x_1 = x_2 = ... = x_{n-1} = x, x_n = n-(n-1)x$. This case can be completed easily as shown in the end of the proof to the example **??**.

∇

Example 1.15.8. *Let $x_1, x_2, ..., x_n$ $(n \geq 4)$ be positive real numbers such that $x_1+x_2+...+x_n = n$. Prove that*

$$\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}+\frac{\sqrt{n}\left(\sqrt{n+4}+2\sqrt{n-1}\right)}{\sqrt{x_1^2+x_2^2+...+x_n^2}} \geq n+2\sqrt{n-1}+\sqrt{n+4}.$$

(Pham Kim Hung)

SOLUTION. Similarly with the previous example, if we fix $x_1+x_2+x_3, x_1^2+x_2^2+x_3^2$ and fix $x_4, x_4, ..., x_n$ then the left hand side of the inequality is a decreasing function of $x_1x_2x_3$. So we may assume that $x_1 = x_2 = ... = x_{n-1}$. For convenience, we may consider the inequality for $n+1$ numbers with the assumption that $x_1 = x_2 = ... = x_n = x$ and $x_{n+1} = n+1-nx$. We have to prove that

$$\frac{n}{x}+\frac{1}{n+1-nx}+\frac{\sqrt{n}\left(\sqrt{n+5}+2\sqrt{n}\right)}{\sqrt{nx^2+(n+1-nx)^2}} \geq n+1+2\sqrt{n}+\sqrt{n+5}.$$

Using the identities

$$\frac{n}{x}+\frac{1}{n+1-nx}-(n+1) = \frac{n(n+1)(1-x)^2}{x(n+1-nx)}\ ;$$

$$nx^2+(n+1-nx)^2-(n+1) = n(n+1)(1-x)^2\ ,$$

our inequality can be transformed to

$$\frac{n(n+1)(1-x)^2}{x(n+1-nx)} \geq \frac{\left(\sqrt{n+5}+2\sqrt{n}\right)n(n+1)(1-x)^2}{\sqrt{(n+1)\left(nx^2+(n+1-nx)^2\right)}+nx^2+(n+1-nx)^2}.$$

It remains to prove that

$$\sqrt{(n+1)\left(nx^2+(n+1-nx)^2\right)}+nx^2+(n+1-nx)^2 \geq \left(\sqrt{n+5}+2\sqrt{n}\right)\cdot x(n+1-nx) \ (\star)$$

By **AM-GM** inequality, we get that

$$nx^2+(n+1-nx)^2 \geq 2\sqrt{n}\cdot x(n+1-nx)\ .$$

By **Holder** inequality, we get that

$$\left(\frac{n}{(n+1-nx)^2}+\frac{1}{x^2}\right)\left((n+1-nx)+(nx)\right)^2 \geq \left(\sqrt[3]{n}+\sqrt[3]{n^2}\right)^3$$

$$\Rightarrow nx^2+(n+1-nx)^2 \geq \frac{n\left(\sqrt[3]{n}+1\right)^3}{(n+1)^2}\cdot x^2(n+1-nx)^2 \geq \frac{n+4}{n+1}\cdot x^2(n+1-nx)^2.$$

These two results combined can deduce $(\star)$ immediately, so we have the desired result. The equality holds for $x_1=x_2=...=x_n=1$.

Comment. 1. For $n=5$, we get the following result

★ *Given positive real numbers a,b,c,d,e with sum 5, prove that*

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}+\frac{1}{e}+\frac{7\sqrt{5}}{\sqrt{a^2+b^2+c^2+d^2+e^2}} \geq 12.$$

2. Another inequality derived from this problem is

★ *Let $x_1,x_2,...,x_n$ be positive real numbers with sum n. Prove that*

$$\frac{1}{x_1}+\frac{1}{x_2}+...+\frac{1}{x_n}+\frac{3\sqrt{n}}{\sqrt{x_1^2+x_2^2+...+x_n^2}} \geq n+3.$$

∇

Article 7

Majorization and Karamata Inequality

1.16 Theory of Majorization

The theory of majorization and convex functions is an important and difficult part of inequalities, with many nice and powerful applications. will discuss in this article is **Karamata** inequality; however, it's necessary to review first some basic properties of majorization.

Definition 1. *Given two sequences $(a) = (a_1, a_2, ..., a_n)$ and $(b) = (b_1, b_2, ..., b_n)$ (where $a_i, b_i \in \mathbb{R}\ \forall i \in \{1, 2, ..., n\}$). We say that the sequence (a) majorizes the sequence (b), and write $(a) \gg (b)$, if the following conditions are fulfilled*

$$\begin{aligned}
&a_1 \geq a_2 \geq ... \geq a_n\ ;\\
&b_1 \geq b_2 \geq ... \geq b_n\ ;\\
&a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_n\ ;\\
&a_1 + a_2 + ... + a_k \geq b_1 + b_2 + ... + b_k\ \forall k \in \{1, 2, ...n-1\}\ .
\end{aligned}$$

Definition 2. *For an arbitrary sequence $(a) = (a_1, a_2, ..., a_n)$, we denote (a^*), a permutation of elements of (a) which are arranged in increasing order: $(a^*) = (a_{i_1}, a_{i_2}, ..., a_{i_n})$ with $a_{i_1} \geq a_{i_2} \geq ... \geq a_{i_n}$ and $\{i_1, i_2, ..., i_n\} = \{1, 2, ..., n\}$.*

Here are some basic properties of sequences.

Proposition 1. *Let $a_1, a_2, ..., a_n$ be real numbers and $a = \dfrac{1}{n}(a_1 + a_2 + ... + a_n)$, then*

$$(a_1, a_2, ..., a_n)^* \gg (a, a, ..., a).$$

Proposition 2. *Suppose that $a_1 \geq a_2 \geq ... \geq a_n$ and $\pi = (\pi_1, \pi_2, ...\pi_n)$ is an arbitrary permutation of $(1, 2, ..., n)$, then we have*

$$(a_1, a_2, ..., a_n) \gg (a_{\pi(1)}, a_{\pi(2)}, ..., a_{\pi(n)}).$$

Proposition 3. *Let $(a) = (a_1, a_2, ..., a_n)$ and $(b) = (b_2, b_2, ..., b_n)$ be two sequences of real numbers. We have that (a^*) majorizes (b) if the following conditions are fulfilled*

$$b_1 \geq b_2 \geq ... \geq b_n \;;$$
$$a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_n \;;$$
$$a_1 + a_2 + ... + a_k \geq b_1 + b_2 + ... + b_k \; \forall k \in \{1, 2, ..., n-1\} \;;$$

These properties are quite obvious: they can be proved directly from the definition of Majorization. The following results, especially the Symmetric Mjorization Criterion, will be most important in what follows.

Proposition 4. *If $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$ are positive real numbers such that $x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n$ and $\dfrac{x_i}{x_j} \geq \dfrac{y_i}{y_j}$ $\forall i < j$, then*

$$(x_1, x_2, ..., x_n) \gg (y_1, y_2, ..., y_n).$$

Proof. To prove this assertion, we will use induction. Because $\dfrac{x_i}{x_1} \leq \dfrac{y_i}{y_1}$ for all $i \in \{1, 2, ..., n\}$, we get that

$$\frac{x_1 + x_2 + ... + x_n}{x_1} \leq \frac{y_1 + y_2 + ... + y_n}{y_1} \Rightarrow x_1 \geq y_1.$$

Consider two sequences $(x_1 + x_2, x_3, ..., x_n)$ and $(y_1 + y_2, y_3, ..., y_n)$. By the inductive hypothesis, we get

$$(x_1 + x_2, x_3, ..., x_n) \gg (y_1 + y_2, y_3, ..., y_n).$$

Combining this with the result that $x_1 \geq y_1$, we have the conclusion immediately.

∇

Theorem 7 (Symmetric Majorization Criterion)**.** *Suppose that $(a) = (a_1, a_2, ..., a_n)$ and $(b) = (b_1, b_2, ..., b_n)$ are two sequences of real numbers; then $(a^*) \gg (b^*)$ if and only if for all real numbers x we have*

$$|a_1 - x| + |a_2 - x| + ... + |a_n - x| \geq |b_1 - x| + |b_2 - x| + ... + |b_n - x|.$$

Proof. To prove this theorem, we need to prove the following.

(i). Necessary condition. Suppose that $(a^*) \gg (b^*)$, then we need to prove that for all real numbers x

$$|a_1 - x| + |a_2 - x| + ... + |a_n - x| \geq |b_1 - x| + |b_2 - x| + ... + |b_n - x| \quad (\star)$$

Notice that $(\star)$ is just a direct application of **Karamata** inequality to the convex function $f(x) = |x - a|$; however, we will prove algebraically.

WLOG, assume that $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$, then $(a) \gg (b)$ by hypothesis. Obviously, $(\star)$ is true if $x \geq b_1$ or $x \leq b_n$, because in these cases, we have

$$\text{RHS} = |b_1 + b_2 + ... + b_n - nx| = |a_1 + a_2 + ... + a_n - nx| \leq \text{LHS}.$$

Consider the case when there exists an integer $k \in \{1, 2, ..., n-1\}$ for which $b_k \geq x \geq b_{k+1}$. In this case, we can remove the absolute value signs of the right-hand expression of $(\star)$

$$|b_1 - x| + |b_2 - x| + ... + |b_k - x| = b_1 + b_2 + ... + b_k - kx \ ;$$
$$|b_{k+1} - x| + |b_{k+2} - x| + ... + |b_n - x| = (n-k)x - b_{k+1} - b_{k+2} - ... - b_n \ ;$$

Moreover, we also have that

$$\sum_{i=1}^{k} |a_i - x| \geq -kx + \sum_{i=1}^{k} a_i,$$

and similarly,

$$\sum_{i=k+1}^{n} |a_i - x| = \sum_{i=k+1}^{n} |x - a_i| \geq (n-k)x - \sum_{i=k+1}^{n} a_i.$$

Combining the two results and noticing that $\sum\limits_{i=1}^{k} a_i \geq \sum\limits_{i=1}^{k} a_i$ and $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} b_i$, we get

$$\sum_{i=1}^{n} |a_i - x| \geq (n-2k)x + \sum_{i=1}^{k} a_i - \sum_{i=k+1}^{n} a_i$$

$$= 2\sum_{i=1}^{k} a_i - \sum_{i=1}^{n} a_i + (n-2k)x \geq 2\sum_{i=1}^{k} b_i - \sum_{i=1}^{n} b_i + (n-2k)x = \sum_{i=1}^{n} |b_i - x|.$$

This last inequality asserts our desired result.

(ii). Sufficient condition. Suppose that the inequality

$$|a_1 - x| + |a_2 - x| + ... + |a_n - x| \geq |b_1 - x| + |b_2 - x| + ... + |b_n - x| \ (\star\star)$$

has been already true for every real number x. We have to prove that $(a^*) \gg (b^*)$.

Without loss of generality, we may assume that $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$. Because $(\star\star)$ is true for all $x \in \mathbb{R}$, if we choose $x \geq \max\{a_i, b_i\}_{i=1}^{n}$ then

$$\sum_{i=1}^{n} |a_i - x| = nx - \sum_{i=1}^{n} a_i \ ; \ \sum_{i=1}^{n} |b_i - x| = nx - \sum_{i=1}^{n} b_i \ ;$$

$$\Rightarrow a_1 + a_2 + ... + a_n \leq b_1 + b_2 + ... + b_n.$$

Similarly, if we choose $x \le \min\{a_i, b_i\}_{i=1}^n$, then

$$\sum_{i=1}^n |a_i - x| = -nx + \sum_{i=1}^n a_i \; ; \; \sum_{i=1}^n |b_i - x| = -nx + \sum_{i=1}^n b_i \; ;$$

$$\Rightarrow a_1 + a_2 + ... + a_n \ge b_1 + b_2 + ... + b_n.$$

From these results, we get that $a_1 + a_2 + ... + a_n = b_1 + b_2 + ... + b_n$. Now suppose that x is a real number in $[a_k, a_{k+1}]$, then we need to prove that $a_1 + a_2 + ... + a_k \ge b_1 + b_2 + ... + b_k$. Indeed, we can eliminate the absolute value signs on the left-hand expression of $(\star\star)$ as follows

$$|a_1 - x| + |a_2 - x| + ... + |a_k - x| = a_1 + a_2 + ... + a_k - kx \;\; ;$$

$$|a_{k+1} - x| + |a_{k+2} - x| + ... + |a_n - x| = (n-k)x - a_{k+1} - a_{k+2} - ... - a_n \;\; ;$$

$$\Rightarrow \sum_{i=1}^n |a_i - x| = (n-2k)x + 2\sum_{i=1}^k a_i - \sum_{i=1}^n a_i.$$

Considering the right-hand side expression of $(\star\star)$, we have

$$\sum_{i=1}^n |b_i - x| = \sum_{i=1}^k |b_i - x| + \sum_{i=k+1}^n |x - b_i|$$

$$\ge -kx + \sum_{i=1}^k b_i + (n-k)x - \sum_{i=k+1}^n |b_i| = (n-2k)x + 2\sum_{i=1}^k |b_i| - \sum_{i=1}^n |b_i|.$$

From these relations and $(\star\star)$, we conclude that

$$(n-2k)x + 2\sum_{i=1}^k a_i - \sum_{i=1}^n a_i \ge (n-2k)x + 2\sum_{i=1}^k |b_i| - \sum_{i=1}^n |b_i|$$

$$\Rightarrow a_1 + a_2 + ... + a_k \ge b_1 + b_2 + ... + b_k,$$

which is exactly the desired result. The proof is completed.

∇

The Symmetric Majorization Criterion asserts that when we examine the majorization of two sequences, it's enough to examine only one conditional inequality which includes a real variable x. This is important because if we use the normal method, there may too many cases to check.

The essential importance of majorization lies in the **Karamata** inequality that which will be discussed right now.

1.17 Karamata Inequality

Karamata inequality is a strong application of convex functions to inequalities. As shown in chapter I, the function f is called convex on $\mathbb{I}$ if and only if $af(x)+bf(y) \geq f(ax+by)$ for all $x, y \in \mathbb{I}$ and for all $a, b \in [0,1]$. Moreover, we also have that f is convex if $f''(x) \geq 0\ \forall x \in \mathbb{I}$. In the following proof of **Karamata** inequality, we only consider a convex function f when $f''(x) \geq 0$ because this case mainly appears in Mathematical Contests. This proof is also a nice application of **Abel** formula.

Theorem 8 (Karamata inequality)**.** *If (a) and (b) two numbers sequences for which $(a^*) \gg (b^*)$ and f is a convex function twice differentiable on $\mathbb{I}$ then*

$$f(a_1)+f(a_2)+...+f(a_n) \geq f(b_1)+f(b_2)+...+f(b_n).$$

PROOF. WLOG, assume that $a_1 \geq a_2 \geq ... \geq a_n$ and $b_1 \geq b_2 \geq ... \geq b_n$. The inductive hypothesis yields $(a) = (a^*) \gg (b^*) = (b)$. Notice that f is a twice differentiable function on $\mathbb{I}$ (that means $f''(x) \geq 0$), so by **Rolle's** theorem, we claim that

$$f(x)-f(y) \geq (x-y)f'(y)\ \forall x, y \in \mathbb{I}.$$

From this result, we also have $f(a_i)-f(b_i) \geq (a_i-b_i)f'(b_i)\ \forall i \in \{1,2,...,n\}$. Therefore

$$\sum_{i=1}^{n} f(a_i) - \sum_{i=1}^{n} f(b_i) = \sum_{i=1}^{n} \left(f(a_i)-f(b_i)\right) \geq \sum_{i=1}^{n}(a_i-b_i)f'(b_i)$$

$$= (a_1-b_1)(f'(b_1)-f'(b_2)) + (a_1+a_2-b_1-b_2)(f'(b_2)-f'(b_3)) + ...+$$

$$+\left(\sum_{i=1}^{n-1} a_i - \sum_{i=1}^{n-1} b_i\right)\left(f'(b_{n-1})-f'(b_n)\right) + \left(\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i\right) f'(b_n) \geq 0$$

because for all $k \in \{1,2,...,n\}$ we have $f'(b_k) \geq f'(b_{k+1})$ and $\sum\limits_{i=1}^{k} a_i \geq \sum\limits_{i=1}^{k} b_i$.

Comment. 1. If f is a non-decreasing function, it is certain that the last condition $\sum\limits_{i=1}^{n} a_i = \sum\limits_{i=1}^{n} b_i$ can be replaced by the stronger one $\sum\limits_{i=1}^{n} a_i \geq \sum\limits_{i=1}^{n} b_i$.

2. A similar result for concave functions is that

> ★ *If $(a) \gg (b)$ are number arrays and f is a concave function twice differentiable then*
>
> $$f(a_1)+f(a_2)+...+f(a_n) \leq f(b_1)+f(b_2)+...+f(b_n).$$

3. If f is convex (that means $\alpha f(a) + \beta f(b) \geq f(\alpha a + \beta b)\ \forall \alpha, \beta \geq 0, \alpha+\beta = 1$) but not twice differentiable ($f''(x)$ does not exist), **Karamata** inequality is still true. A detailed proof can be seen in the book **Inequalities** written by G.H Hardy, J.E Littewood and G.Polya.

∇

The following examples should give you a sense of how this inequality can be used.

Example 1.17.1. *If f is a convex function then*

$$f(a)+f(b)+f(c)+f\left(\frac{a+b+c}{3}\right)\geq\frac{4}{3}\left(f\left(\frac{a+b}{2}\right)+f\left(\frac{b+c}{2}\right)+f\left(\frac{c+a}{2}\right)\right).$$

(Popoviciu's inequality)

SOLUTION. WLOG, suppose that $a\geq b\geq c$. Consider the following number sequences

$$(x)=(a,a,a,b,t,t,t,b,b,c,c,c)\quad;\quad(y)=(\alpha,\alpha,\alpha,\alpha,\beta,\beta,\beta,\beta,\gamma,\gamma,\gamma,\gamma)\quad;$$

where

$$t=\frac{a+b+c}{3}\ ,\ \alpha=\frac{a+b}{2}\ ,\ \beta=\frac{a+c}{2}\ ,\ \gamma=\frac{b+c}{2}\ .$$

Clearly, we have that (y) is a monotonic sequence. Moreover

$$a\geq\alpha,3a+b\geq4\alpha,3a+b+t\geq4\alpha+2\beta,3a+b+3t\geq4\alpha+3\beta,$$

$$3a+2b+3t\geq4\alpha+4\beta,3a+3b+3t\geq4\alpha+4\beta+\gamma,$$

$$3a+3b+3t+c\geq4\alpha+4\beta+2\gamma,3a+3b+3t+3c\geq4\alpha+4\beta+4\gamma.$$

Thus $(x^*)\gg(y)$ and therefore $(x^*)\gg(y^*)$. By **Karamata** inequality, we conclude

$$3\left(f(x)+f(y)+f(z)+f(t)\right)\geq4\left(f(\alpha)+f(\beta)+f(\gamma)\right),$$

which is exactly the desired result. We are done.

∇

Example 1.17.2 (Jensen Inequality)**.** *If f is a convex function then*

$$f(a_1)+f(a_2)+...+f(a_n)\geq nf\left(\frac{a_1+a_2+...+a_n}{n}\right).$$

SOLUTION. We use property 1 of majorization. Suppose that $a_1\geq a_2\geq...\geq a_n$, then we have $(a_1,a_2,...,a_n)\gg(a,a,...,a)$ with $a=\frac{1}{n}(a_1+a_2+...+a_n)$. Our problem is directly deduced from **Karamata** inequality for these two sequences.

∇

Example 1.17.3. *Let a,b,c,x,y,z be six real numbers in $\mathbb{I}$ satisfying*

$$a+b+c=x+y+z,\max(a,b,c)\geq\max(x,y,z),\min(a,b,c)\leq\min(x,y,z),$$

then for every convex function f on $\mathbb{I}$, we have

$$f(a)+f(b)+f(c)\geq f(x)+f(y)+f(z).$$

SOLUTION. Assume that $x \geq y \geq z$. The assumption implies $(a,b,c)^* \gg (x,y,z)$ and the conclusion follows from **Karamata** inequality.

$$\nabla$$

Example 1.17.4. *Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that*

$$(1+a_1)(1+a_2)...(1+a_n) \leq \left(1+\frac{a_1^2}{a_2}\right)\left(1+\frac{a_2^2}{a_3}\right)...\left(1+\frac{a_n^2}{a_1}\right).$$

SOLUTION. Our inequality is equivalent to

$$\ln(1+a_1)+\ln(1+a_2)+...+\ln(1+a_n) \leq \ln\left(1+\frac{a_1^2}{a_2}\right)+\ln\left(1+\frac{a_2^2}{a_3}\right)+...+\ln\left(1+\frac{a_n^2}{a_1}\right).$$

Suppose that the number sequence $(b) = (b_1, b_2, ..., b_n)$ is a permutation of $(\ln a_1, \ln a_2, ..., \ln a_n)$ which was rearranged in decreasing order. We may assume that $b_i = \ln a_{k_i}$, where $(k_1, k_2, ..., k_n)$ is a permutation of $(1, 2, .., n)$. Therefore the number sequence $(c) = (2\ln a_1 - \ln a_2, 2\ln a_2 - \ln a_3, ..., 2\ln a_n - \ln a_1)$ can be rearranged into a new one as

$$(c') = (2\ln a_{k_1} - \ln a_{k_1+1}, 2\ln a_{k_2} - \ln a_{k_2+1}, ..., 2\ln a_{k_n} - \ln a_{k_n+1}).$$

Because the number sequence $(b) = (\ln a_{k_1}, \ln a_{k_2}, ..., \ln a_{k_n})$ is decreasing, we must have $(c')^* \gg (b)$. By **Karamata** inequality, we conclude that for all convex function x then

$$f(c_1)+f(c_2)+...+f(c_n) \geq f(b_1)+f(b_2)+...+f(b_n),$$

where $c_i = 2\ln a_{k_i} - \ln a_{k_i+1}$ and $b_i = \ln a_{k_i}$ for all $i \in \{1, 2, ..., n\}$. Choosing $f(x) = \ln(1+e^x)$, we have the desired result.

Comment. 1. A different choice of $f(x)$ can make a different problem. For example, with the convex function $f(x) = \sqrt{1+e^x}$, we get

$$\sqrt{1+a_1}+\sqrt{1+a_2}+...+\sqrt{1+a_n} \leq \sqrt{1+\frac{a_1^2}{a_2}}+\sqrt{1+\frac{a_2^2}{a_3}}+...+\sqrt{1+\frac{a_n^2}{a_1}}.$$

2. By **Cauchy-Schwarz** inequality, we can solve this problem according to the following estimation

$$\left(1+\frac{a_1^2}{a_2}\right)(1+a_2) \geq (1+a_1)^2.$$

$$\nabla$$

Example 1.17.5. *Let $a_1, a_2, ..., a_n$ be positive real numbers. Prove that*

$$\frac{a_1^2}{a_2^2+...+a_n^2}+...+\frac{a_n^2}{a_1^2+...+a_{n-1}^2} \geq \frac{a_1}{a_2+...+a_n}+...+\frac{a_n}{a_1+...+a_{n-1}}.$$

SOLUTION. For each $i \in \{1, 2, ..., n\}$, we denote

$$y_i = \frac{a_i}{a_1 + a_2 + ... + a_n},\ x_i = \frac{a_i^2}{a_1^2 + a_2^2 + ... + a_n^2}$$

then $x_1 + x_2 + ... + x_n = y_1 + y_2 + ... + y_n = 1$. We need to prove that

$$\sum_{i=1}^{n} \frac{x_i}{1 - x_i} \geq \sum_{i=1}^{n} \frac{y_i}{1 - y_i}.$$

WLOG, assume that $a_1 \geq a_2 \geq ... \geq a_n$, then certainly $x_1 \geq x_2 \geq ... \geq x_n$ and $y_1 \geq y_2 \geq ... \geq y_n$. Moreover, for all $i \geq j$, we also have

$$\frac{x_i}{x_j} = \frac{a_i^2}{a_j^2} \geq \frac{a_i}{a_j} = \frac{y_i}{y_j}.$$

By property 4, we deduce that $(x_1, x_2, ..., x_n) \gg (y_1, y_2, ..., y_n)$. Furthermore,

$$f(x) = \frac{x}{1 - x}$$

is a convex function, so by **Karamata** inequality, the final result follows immediately.

∇

Example 1.17.6. *Suppose that $(a_1, a_2, ..., a_{2n})$ is a permutation of $(b_1, b_2, ..., b_{2n})$ which satisfies $b_1 \geq b_2 \geq ... \geq b_{2n} \geq 0$. Prove that*

$$(1 + a_1a_2)(1 + a_3a_4)...(1 + a_{2n-1}a_{2n})$$

$$\leq (1 + b_1b_2)(1 + b_3b_4)...(1 + b_{2n-1}b_{2n}).$$

SOLUTION. Denote $f(x) = \ln(1 + e^x)$ and $x_i = \ln a_i, y_i = \ln b_i$. We need to prove that

$$f(x_1 + x_2) + f(x_3 + x_4) + ... + f(x_{2n-1} + x_{2n})$$

$$\leq f(y_1 + y_2) + f(y_3 + y_4) + ... + f(y_{2n-1} + y_{2n}).$$

Consider the number sequences $(x) = (x_1 + x_2, x_3 + x_4, ..., x_{2n-1} + x_{2n})$ and $(y) = (y_1 + y_2, y_3 + y_4, ..., y_{2n-1} + y_{2n})$. Because $y_1 \geq y_2 \geq ... \geq y_n$, if $(x^*) = (x_1^*, x_2^*, ..., x_n^*)$ is a permutation of elements of (x) which are rearranged in the decreasing order, then

$$y_1 + y_2 + ... + y_{2k} \geq x_1^* + x_2^* + ... + x_{2k}^*,$$

and therefore $(y) \gg (x^*)$. The conclusion follows from **Karamata** inequality with the convex function $f(x)$ and two numbers sequences $(y) \gg (x^*)$.

∇

If these examples are just the beginner's applications of **Karamata** inequality, you will see much more clearly how effective this theorem is in combination with the Symmetric Majorization Criterion. Famous Turkevici's inequality is such an instance.

Example 1.17.7. *Let a, b, c, d be non-negative real numbers. Prove that*

$$a^4 + b^4 + c^4 + d^4 + 2abcd \geq a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2.$$

(Turkevici's inequality)

SOLUTION. To prove this problem, we use the following lemma

★ *For all real numbers x, y, z, t then*

$$2(|x|+|y|+|z|+|t|)+|x+y+z+t| \geq |x+y|+|y+z|+|z+t|+|t+x|+|x+z|+|y+t|.$$

We will not give a detailed proof of this lemma now (because the next problem shows a nice generalization of this one, with a meticulous solution). At this time, we will clarify that this lemma, in combination with **Karamata** inequality, can directly give Turkevici's inequality. Indeed, let $a = e^{a_1}, b = e^{b_1}, c = e^{c_1}$ and $d = e^{d_1}$, our problem is

$$\sum_{cyc} e^{4a_1} + 2e^{a_1+b_1+c_1+d_1} \geq \sum_{sym} e^{2a_1+2b_1}.$$

Because $f(x) = e^x$ is convex, it suffices to prove that (a^*) majorizes (b^*) with

$$(a) = (4a_1, 4b_1, 4c_1, 4d_1, a_1 + b_1 + c_1 + d_1, a_1 + b_1 + c_1 + d_1)\ ;$$

$$(b) = (2a_1 + 2b_1, 2b_1 + 2c_1, 2c_1 + 2d_1, 2d_1 + 2a_1, 2a_1 + 2c_1, 2b_1 + 2d_1)\ ;$$

By the symmetric majorization criterion, we need to prove that for all $x_1 \in \mathbb{R}$ then

$$2|a_1 + b_1 + c_1 + d_1 - 4x_1| + \sum_{cyc} |4a_1 - 4x_1| \geq \sum_{sym} |2a_1 + 2b_1 - 4x_1|.$$

Letting now $x = a_1 - x_1, y = b_1 - x_1, z = c_1 - x_1, t = d_1 - x_1$, we obtain an equivalent form as

$$2\sum_{cyc} |x| + |\sum_{cyc} x| \geq \sum_{sym} |x + y|,$$

which is exactly the lemma shown above. We are done.

∇

Example 1.17.8. *Let $a_1, a_2, ..., a_n$ be non-negative real numbers. Prove that*

$$(n-1)(a_1^2+a_2^2+...+a_n^2)+n\sqrt[n]{a_1^2a_2^2...a_n^2} \geq (a_1+a_2+...+a_n)^2.$$

SOLUTION. We realize that Turkevici's inequality is a particular case of this general problem (for $n=4$, it becomes Turkevici's). By using the same reasoning as in the preceding problem, we only need to prove that for all real numbers $x_1, x_2, ..., x_n$ then $(a^*) \gg (b^*)$ with

$$(a) = (\underbrace{2x_1, 2x_1, ..., 2x_1}_{n-1}, \underbrace{2x_2, 2x_2, ..., 2x_2}_{n-1}, ..., \underbrace{2x_n, 2x_n, ..., 2x_n}_{n-1}, \underbrace{2x, 2x, ..., 2x}_{n})\ ;$$

$$(b) = (x_1+x_1, x_1+x_2, x_1+x_3, ..., x_1+x_n, x_2+x_1, x_2+x_2, ..., x_2+x_n, ..., x_n+x_n)\ ;$$

and $x = \dfrac{1}{n}(x_1+x_2+...+x_n)$. By the Symmetric Majorization Criterion, it suffices to prove that

$$(n-2)\sum_{i=1}^{n}|x_i| + |\sum_{i=1}^{n}x_i| \geq \sum_{i<j}^{n}|x_i+x_j|.$$

Denote $A = \{i \mid x_i \geq 0\}, B = \{i \mid x_i < 0\}$ and suppose that $|A| = m, |B| = k = n-m$. We will prove an equivalent form as follows: if $x_i \geq 0\ \forall i \in \{1,2,...,n\}$ then

$$(n-2)\sum_{i\in A,B}x_i + |\sum_{i\in A}x_i - \sum_{j\in B}x_j| \geq \sum_{(i,j)\in A,B}(x_i+x_j) + \sum_{i\in A, j\in B}|x_i-x_j|.$$

Because $k+m=n$, we can rewrite the inequality above into

$$(k-1)\sum_{i\in A}x_i + (m-1)\sum_{j\in B}x_j + |\sum_{i\in A}x_i - \sum_{j\in B}x_j| \geq \sum_{i\in A, j\in B}|x_i-x_j|\ (\star)$$

Without loss of generality, we may assume that $\sum\limits_{i\in A}x_i \geq \sum\limits_{j\in B}x_j$. For each $i \in A$, let $|A_i| = \{j \in B | x_i \leq x_j\}$ and $r_i = |A_i|$. For each $j \in B$, let $|B_j| = \{i \in A | x_j \leq x_i\}$ and $s_j = |B_j|$. Thus the left-hand side expression in $(\star)$ can be rewritten as

$$\sum_{i\in A}(k-2r_i)x_i + \sum_{j\in B}(m-2s_j)x_j.$$

Therefore $(\star)$ becomes

$$\sum_{i\in A}(2r_i-1)x_i + \sum_{j\in B}(2s_j-1)x_j + |\sum_{i\in A}x_i - \sum_{j\in B}x_j| \geq 0$$

$$\Leftrightarrow\ \sum_{i\in A}r_ix_i + \sum_{j\in B}(s_j-1)x_j \geq 0.$$

Notice that if $s_j \geq 1$ for all $j \in \{1,2,...,n\}$ then we have the desired result immediately. Otherwise, assume that there exists a number $s_l = 0$, then

$$\max_{i\in A\cup B}x_i \in B\ \Rightarrow r_i \geq 1\ \forall i \in \{1,2,...,m\}.$$

Thus

$$\sum_{i\in A} r_i x_i + \sum_{j\in B}(s_j-1)x_j \geq \sum_{i\in A} x_i - \sum_{j\in B} x_j \geq 0.$$

This problem is completely solved. The equality holds for $a_1 = a_2 = ... = a_n$ and $a_1 = a_2 = ... = a_{n-1}, a_n = 0$ up to permutation.

∇

Example 1.17.9. *Let $a_1, a_2, ..., a_n$ be positive real numbers with product 1. Prove that*

$$a_1 + a_2 + ... + a_n + n(n-2) \geq (n-1)\left(\frac{1}{\sqrt[n-1]{a_1}} + \frac{1}{\sqrt[n-1]{a_2}} + ... + \frac{1}{\sqrt[n-1]{a_n}}\right).$$

SOLUTION. The inequality can be rewritten in the form

$$\sum_{i=1}^n a_i + n(n-1)\sqrt[n]{\prod_{i=1}^n a_i} \geq (n-1)\sum_{i=1}^n \sqrt[n-1]{\prod_{j\neq i} a_j}.$$

First we will prove the following result (that helps us prove the previous inequality immediately): if $x_1, x_2, ..., x_n$ are real numbers then $(\alpha^*) \gg (\beta^*)$ with

$$(\alpha) = (x_1, x_2, ..., x_n, x, x, ..., x)\ \ ;$$

$$(\beta) = (y_1, y_1, ..., y_1, y_2, y_2, ..., y_2, ..., y_n, y_n, ..., y_n)\ \ ;$$

where $x = \dfrac{1}{n}(x_1 + x_2 + ... + x_n)$, (α) includes $n(n-2)$ numbers x, (β) includes $n-1$ numbers y_k $(\forall k \in \{1, 2, ..., n\})$, and each number b_k is determined from $b_k = \dfrac{nx - x_i}{n-1}$.

Indeed, by the symmetric majorization criterion, we only need to prove that

$$|x_1| + |x_2| + ... + |x_n| + (n-2)|S| \geq |S - x_1| + |S - x_2| + ... + |S - x_n|\ (\star)$$

where $S = x_1 + x_2 + ... + x_n = nx$. In case $n = 3$, this becomes a well-known result

$$|x| + |y| + |z| + |x + y + z| \geq |x + y| + |y + z| + |z + x|.$$

In the general case, assume that $x_1 \geq x_2 \geq ... \geq x_n$. If $x_i \geq S\ \forall i \in \{1, 2, ..., n\}$ then

$$\text{RHS} = \sum_{i=1}^n (x_i - S) = -(n-1)S \leq (n-1)|S| \leq \sum_{i=1}^n |x_i| + (n-2)|S| = \text{LHS}.$$

and the conclusion follows. Case $x_i \leq S\ \forall i \in \{1, 2, ..., n\}$ is proved similarly. We consider the final case. There exists an integer k $(1 \leq k \leq n-1)$ such that $x_k \geq S \geq x_{k+1}$. In this case, we can prove $(\star)$ simply as follows

$$\text{RHS} = \sum_{i=1}^k (x_i - S) + \sum_{i=k+1}^n (S - x_i) = \sum_{i=1}^k x_i - \sum_{i=k+1}^n x_{k+1} + (n-2k)S,$$

$$\leq \sum_{i=1}^{n} |x_i| + (n-2k)|S| \leq \sum_{i=1}^{n} |x_i| + (n-2)|S| = \text{LHS},$$

which is also the desired result. The problem is completely solved.

∇

Example 1.17.10. *Let $a_1, a_2, ..., a_n$ be non-negative real numbers. Prove that*

$$(n-1)\left(a_1^n + a_2^n + ... + a_n^n\right) + na_1a_2...a_n \geq (a_1+a_2+...+a_n)\left(a_1^{n-1} + a_2^{n-1} + ... + a_n^{n-1}\right).$$

(Suranji's inequality)

SOLUTION. We will prove first the following result for all real numbers $x_1, x_2, ..., x_n$

$$n(n-1)\sum_{i=1}^{n} |x_i| + n|S| \geq \sum_{i,j=1}^{n} |x_i + (n-1)x_j| \quad (1)$$

in which $S = x_1 + x_2 + ... + x_n$. Indeed, let $z_i = |x_i|\ \forall i \in \{1, 2, ..., n\}$ and $A = \{i \,\big|\, 1 \leq i \leq n, i \in \mathbb{N}, x_i \geq 0\}$, $B = \{i \,\big|\, 1 \leq i \leq n, i \in \mathbb{N}, x_i < 0\}$. WLOG, we may assume that $A = \{1, 2, ..., k\}$ and $B = \{k+1, k+2, ..., n\}$, then $|A| = k, |B| = n - k = m$ and $z_i \geq 0$ for all $i \in A \cup B$. The inequality above becomes

$$n(n-1)\left(\sum_{i\in A} z_i + \sum_{j\in B} z_j\right) + n\left|\sum_{i\in A} z_i - \sum_{j\in B} z_j\right|$$

$$\geq \sum_{i,i'\in A} |z_i + (n-1)z_{i'}| + \sum_{j,j'\in B} |z_j + z_{j'}| + \sum_{i\in A, j\in B} \big(|z_i - (n-1)z_j| + |(n-1)z_i - z_j|\big)$$

Because $n = k + m$, the previous inequality is equivalent to

$$n(m-1)\sum_{i\in A} z_i + n(k-1)\sum_{j\in B} z_j + n\left|\sum_{i\in A} z_i - \sum_{j\in B} z_j\right|$$

$$\geq \sum_{i\in A, j\in B} |z_i - (n-1)z_j| + \sum_{i\in A, j\in B} |(n-1)z_i - z_j| \quad (\star)$$

For each $i \in A$ we denote

$$B_i = \{\, j \in B \big| (n-1)z_i \geq z_j \,\}\ ;\ B'_i = \{j \in B \big| z_i \geq (n-1)z_j\}\ ;$$

For each $j \in B$ we denote

$$A_j = \{\, i \in A \big| (n-1)z_j \geq z_i \,\}\ ;\ A'_j = \{i \in A \big| z_j \geq (n-1)z_i\}\ ;$$

We have of course $B'_i \subset B_i \subset B$ and $A'_i \subset A_i \subset A$. After giving up the absolute value signs, the right-hand side expression of $(\star)$ is indeed equal to

$$\sum_{i\in A} \left(mn - 2|B'_i| - 2(n-1)|B_i|\right) z_i + \sum_{j\in B} \left(kn - 2|A'_j| - 2(n-1)|A_j|\right) z_j.$$

WLOG, we may assume that $\sum\limits_{i\in A} z_i \geq \sum\limits_{j\in B} z_j$. The inequality above becomes

$$\sum_{i\in A} \left(|B_i'| + (n-1)|B_i|\right) z_i + \sum_{j\in B} \left(|A_j'| + (n-1)|A_j| - n\right) z_j \geq 0.$$

Notice that if for all $j \in B$, we have $|A_j'| \geq 1$, then the conclusion follows immediately (because $A_j' \subset A_j$, then $|A_j| \geq 1$ and $|A_j'| + (n-1)|A_j| - n \geq 0\ \forall j \in B$). If not, we may assume that there exists a certain number $r \in B$ for which $|A_r'| = 0$, and therefore $|A_r| = 0$. Because $|A_r| = 0$, it follows that $(n-1)z_r \leq z_i$ for all $i \in A$. This implies that $|B_i| \geq |B_i'| \geq 1$ for all $i \in A$, therefore $|B_i'| + (n-1)|B_i| \geq n$ and we conclude that

$$\sum_{i\in A} \left(|B_i'| + (n-1)|B_i|\right) z_i + \sum_{j\in B} \left(|A_j'| + (n-1)|A_j| - n\right) z_j \geq n\sum_{i\in A} z_i - n\sum_{j\in B} z_j \geq 0.$$

Therefore (1) has been successfully proved and therefore Suranji's inequality follows immediately from **Karamata** inequality and the Symmetric Majorization Criterion.

∇

Example 1.17.11. *Let $a_1, a_2, ..., a_n$ be positive real numbers such that $a_1 \geq a_2 \geq ... \geq a_n$. Prove the following inequality*

$$\frac{a_1+a_2}{2} \cdot \frac{a_2+a_3}{2} \cdots \frac{a_n+a_1}{2} \leq \frac{a_1+a_2+a_3}{3} \cdot \frac{a_2+a_3+a_4}{3} \cdots \frac{a_n+a_1+a_2}{3}.$$

(V. Adya Asuren)

SOLUTION. By using **Karamata** inequality for the concave function $f(x) = \ln x$, we only need to prove that the number sequence (x^*) majorizes the number sequence (y^*) in which $(x) = (x_1, x_2, ..., x_n)$, $(y) = (y_1, y_2, ..., y_n)$ and for each $i \in \{1, 2, ..., n\}$

$$x_i = \frac{a_i + a_{i+1}}{2},\ y_i = \frac{a_i + a_{i+1} + a_{i+2}}{3}$$

(with the common notation $a_{n+1} = a_1$ and $a_{n+2} = a_2$). According to the Symmetric Majorization Criterion, it suffices to prove the following inequality

$$3\left(\sum_{i=1}^{n} |z_i + z_{i+1}|\right) \geq 2\left(\sum_{i=1}^{n} |z_i + z_{i+1} + z_{i+2}|\right)\ (\star)$$

for all real numbers $z_1 \geq z_2 \geq ... \geq z_n$ and $z_{n+1},\ z_{n+2}$ stand for $z_1,\ z_2$ respectively.

Notice that $(*)$ is obviously true if $z_i \geq 0$ for all $i = 1, 2, ..., n$. Otherwise, assume that $z_1 \geq z_2 \geq ... \geq z_k \geq 0 > z_{k+1} \geq ... \geq z_n$. We realize first that it's enough to consider $(\star)$ for 8 numbers (instead of n numbers). Now consider it for 8 numbers $z_1, z_2, ..., z_8$. For each number $i \in \{1, 2, ..., 8\}$, we denote $c_i = |z_i|$, then $c_i \geq 0$. To prove this problem, we will prove first the most difficult case $z_1 \geq z_2 \geq z_3 \geq z_4 \geq 0 \geq z_5 \geq z_6 \geq z_7 \geq z_8$. Giving up the absolute value signs, the problem becomes

$$3(c_1 + 2c_2 + 2c_3 + c_4 + c_5 + 2c_6 + 2c_7 + c_8 + |c_4 - c_5| + |c_8 - c_1|)$$

$$\geq 2(c_1+2c_2+2c_3+c_4+|c_3+c_4-c_5|+|c_4-c_5-c_6|+c_5+2c_6+2c_7+c_8+|c_7+c_8-c_1|+|c_8-c_1-c_2|)$$

$$\Leftrightarrow \ c_1 + 2c_2 + 2c_3 + c_4 + c_5 + 2c_6 + 2c_7 + c_8 + 3|c_4 - c_5| + 3|c_8 - c_1|$$

$$\geq 2|c_3 + c_4 - c_5| + 2|c_4 - c_5 - c_6| + 2|c_7 + c_8 - c_1| + 2|c_8 - c_1 - c_2|$$

Clearly, this inequality is obtained by adding the following results

$$\begin{aligned} 2|c_4 - c_5| + 2c_3 &\geq 2|c_3 + c_4 + c_5| \\ 2|c_8 - c_1| + 2c_7 &\geq 2|c_7 + c_8 - c_1| \\ |c_4 - c_5| + c_4 + c_5 + 2c_6 &\geq 2|c_4 - c_5 - c_6| \\ |c_8 - c_1| + c_8 + c_1 + 2c_2 &\geq 2|c_8 - c_1 - c_2| \end{aligned}$$

For other cases when there exist exactly three (or five); two (or six); only one (or seven) non-negative numbers in $\{z_1, z_2, ..., z_8\}$, the problem is proved completely similarly (indeed, notice that, for example, if $z_1 \geq z_2 \geq z_3 \geq 0 \geq z_4 \geq z_5 \geq z_6 \geq z_7 \geq z_8$ then we only need to consider the similar but simpler inequality of seven numbers after eliminating z_6). Therefore $(\star)$ is proved and the conclusion follows immediately.

∇

Using **Karamata** inequality together with the theory of majorization like we have just done it is an original method for algebraic inequalities. By this method, a purely algebraic problem can be transformed to a linear inequality with absolute signs, which is essentially an arithmetic problem, and which can have many original solutions.